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ABSTRACT

An extension of a computer-assisted instruction (CAI) program developed at Pennsylvania State University, this handbook provides a variety of suggestions and activities for the teacher to use in the mathematics class. This first part concentrates on exercises designed to help the teacher learn the content, and it is keyed to a textbook and to on-line practice chapters of the program. Topics covered are sets and early number experiences, exponents, the Hindu-Arabic system, other numeration systems, addition, subtraction, multiplication, divisions, integers, fractions, decimals, and ratio and percent. EM 011 037 through EM 011 043, EM 011 046, EM 011 047, and EM 011 049 through EM 011 058 are related documents. (SH)

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COMPUTER ASSISTED INSTRUCTION LABORATORY

COLLEGE OF EDUCATION · CHAMBERS BUILDING

**THE PENNSYLVANIA
STATE UNIVERSITY · UNIVERSITY PARK, PA.**

TEACHER'S HANDBOOK
ELEMENTARY SCHOOL MATHEMATICS

PART I
HELP TO YOU IN LEARNING MATHEMATICS

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ROY F. SHORTT

JUNE 1969

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Note to accompany the Penn State
Documents.

In order to have the entire collection of reports generated by the Computer Assisted Instruction Lab. at Penn State University included in the ERIC archives, the ERIC Clearinghouse on Educational Media and Technology was asked by Penn State to input the material. We are therefore including some documents which may be several years old. Also, so that our bibliographic information will conform with Penn State's, we have occasionally changed the title somewhat, or added information that may not be on the title page. Two of the documents in the CARE (Computer Assisted Remedial Education) collection were transferred to ERIC/EC to abstract. They are Report Number R-36 and Report Number R-50.

Jack C. Groll: ERIC/EC

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INTRODUCTION

This handbook is an extension of the CAI materials with which you will be working. In it you will find a variety of suggestions and activities to be used in your mathematics class. Part I includes exercises to help you in learning mathematics. Part II includes teaching materials, activities for using the materials, illustrative lessons, and games to develop children's understanding and skills. The appendices include reference materials for teachers. You are encouraged to extend and adapt any ideas presented in this manual to meet the needs of your class.

Before going on-line for each chapter, you should read the appropriate chapter indicated below in your textbook, Guiding Discovery in Elementary School Mathematics, by C. Alan Riedesel. The material in Chapter 13 (Ratio and Percent) will be based entirely upon your comprehension of Chapter 10 in the textbook; therefore, you must read this chapter before going on-line.

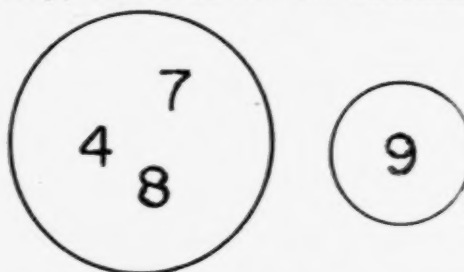
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PART I: Help to You in Learning Mathematics

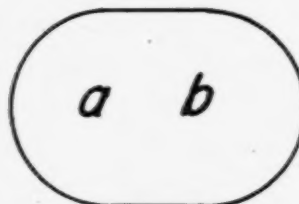
Although much of the theory of sets originated with the German mathematician Georg Cantor (1845-1918), many of the notions of set theory were investigated earlier by the English mathematician George Boole (1815-1864).

In mathematics, a collection of things or objects is called a *set*. The things or objects are the *elements* or *members* of the set. Some examples are: The set of all polygons (a set of *geometric figures*); the set of whole numbers less than 500 (a set of *numbers*); The set whose members are the letters x and y (a set of *letters*). The word set is sometimes used in non-mathematical situations (e.g., a set of dishes, a set of golf clubs, etc.). There are situations where the word set could be used but where other words are preferred (e.g., a herd of cows, a flock of sheep, etc.). We shall continue to use words such as herd and flock in such situations, but in mathematics, we shall use the word set.

Certain sets are easily visualized by using diagrams (often called *Venn diagrams* in honor of the English mathematician John Venn [1834-1883]). In such diagrams the elements, or more properly, the names for the elements, are shown enclosed by simple closed curves. For example, the diagram at the right shows one set which has as its members the numbers 4, 7, and 8 and another set which has only one member, the number 9.



Describing sets: Sometimes a set is described by listing its members within braces. For example, $\{a, b\}$ (read *the set of a and b* or *the set whose members are a and b*) describes the set shown at the right. $\{b, a\}$ also describes this set. The order in which the members are listed is irrelevant. This method of describing a set is called the *listing method* or *roster method*. The set of counting numbers is the set whose members are 1, 2, 3, 4, 5, and so on. We show this by writing $\{1, 2, 3, 4, 5, \dots\}$.



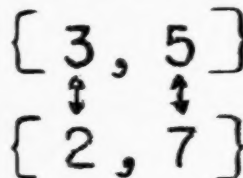
The three dots (...) mean "and so on." The notation $\{1, 2, 3, \dots, 15\}$ means "the set of 1, 2, 3, and so on up to, and including 15." The set of counting numbers is an *infinite* set while $\{1, 2, 3, \dots, 15\}$ is a *finite* set.

Usually, capital letters are used to denote sets. The members of a set are often represented by lower case letters.

Set membership: The symbol $\in$ is used to denote set membership. If a is an element of B , then we write $a \in B$ which is read *a is an element of B* or *a is a member of B*. Thus, $5 \in \{3, 5\}$ and $8 \in \{1, 2, 3, \dots, 10\}$. If a is not an element of B , then we write $a \notin B$ which is read *a is not an element of B* or *a is not a member of B*. Thus, $2 \notin \{3, 5\}$ and $15 \notin \{1, 2, 3, \dots, 10\}$.

Equal sets: Sets which have the same members, that is, which are identical are said to be *equal*. If A denotes $\{3, 5\}$ and B denotes $\{5, 3\}$, then $A = B$. (Remember: The order in which the members are listed is irrelevant.)

Equivalent sets: At the right, the members of $\{3, 5\}$ are shown matched, or paired, with the members of $\{2, 7\}$ such that each member of $\{3, 5\}$ is paired with one and only one member of $\{2, 7\}$ and each member of $\{2, 7\}$ is paired with one and only one member of $\{3, 5\}$. Such a pairing shows that a *one-to-one correspondence* exists between the two sets. When such a correspondence exists between sets, the sets are said to be *equivalent*. If two finite sets each have the same number of elements, then they are equivalent. All sets which are equal are also equivalent, but there exist equivalent sets which are not equal.



The empty set: The set which has no members at all (e.g., the set of elephants that can fly) is called the *empty set*. In grades K-6 the symbol $\{ \}$ is often used to denote the empty set. In more advanced work the symbol $\emptyset$ is used and the empty set is then referred to as the *null set*. There is only one empty set. That is why we refer to *the* empty set. Note that $\{0\}$ does not denote the empty set because $\{0\}$ denotes the set which has 0 as its one member. (Remember: The empty set has no members.)

Disjoint sets: Two sets which have no members in common are said to be *disjoint*. Consider the sets named at the right. Sets A and B are disjoint and sets B and C are disjoint. However, sets A and C are not disjoint (they have at least one common member, 5).

$$A = \{2, 5\}$$

$$B = \{3, 6, 7\}$$

$$C = \{4, 5, 8\}$$

Union of sets: The *union* of two sets is the set of all things that are members of either set or of both sets. The symbol $\cup$ is used to denote the union of sets A and B by writing $A \cup B$ which is read *the union of sets A and B*. If $A = \{2, 3, 5\}$ and $B = \{4, 5, 6, 7\}$, then $A \cup B = \{2, 3, 4, 5, 6, 7\}$.

The union of the set of odd counting numbers and the set of even counting numbers is the set of all counting numbers. That is,

$$\{1, 3, 5, \dots\} \cup \{2, 4, 6, \dots\} = \{1, 2, 3, 4, 5, 6, \dots\}.$$

Also, if A denotes any set, then $A \cup A = A$ and $\{ \} \cup A = A$.

The diagram in Box I at the right suggests that sets A and B are not disjoint. The shaded region in Box II represents $A \cup B$.

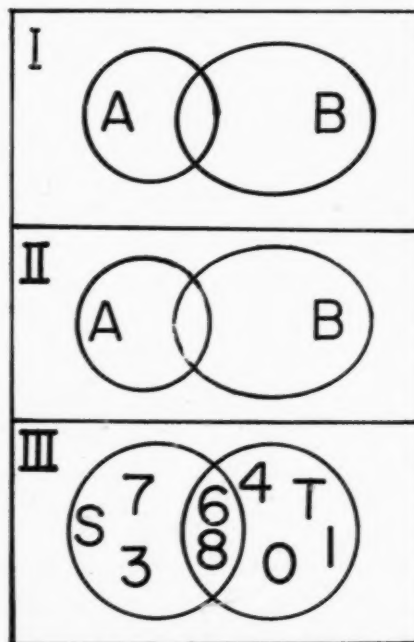
Box III: $S = \{3, 6, 7, 8\}$,
 $T = \{0, 1, 4, 6, 8\}$, and
 $S \cup T = \{0, 1, 3, 4, 6, 7, 8\}$.

Intersection of sets: The *intersection* of two sets is the set of all things that are members of both sets. The symbol $\cap$ is used to denote the intersection of sets A and B by writing $A \cap B$ which is read *the intersection of sets A and B*. For example, if $A = \{2, 3, 5\}$ and $B = \{4, 5, 6, 7\}$, then $A \cap B = \{5\}$.

The intersection of the set of odd counting numbers and the set of even counting numbers is the empty set. That is,

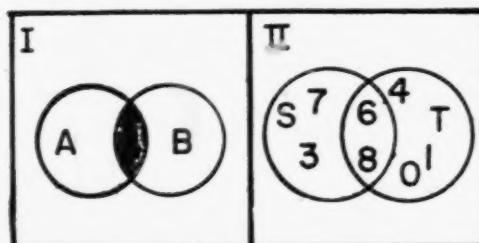
$$\{1, 3, 5, \dots\} \cap \{2, 4, 6, \dots\} = \{ \}.$$

Also, if A denotes any set, then $A \cap A = A$ and $\{ \} \cap A = \{ \}$.



Box I: The shaded region represents $A \cap B$. Box II: $S \cap T = \{6, 8\}$

Since disjoint sets are sets which have no members in common, we can say that two sets are disjoint if the intersection of the two sets is the empty set.



Subset of a set: A set A is a *subset* of a set B if everything that is a member of A is also a member of B. In other words, A does not contain any elements which are not elements of B. The symbol $\subseteq$ is used to write $A \subseteq B$ which is read *A is a subset of B*.

Consider $A = \{3, 7\}$ and $B = \{2, 3, 4, 5, 6, 7\}$. Every member of A is also a member of B so $A \subseteq B$. However, $B \not\subseteq A$ -- that is, B is not a subset of A. Some other subsets of B are $\{\}$, $\{2, 5\}$, $\{2, 3, 6\}$, and so on.

The empty set is a subset of every set. For example, $\{\} \subseteq \{5, 8\}$. If this is troublesome, think: Does $\{\}$ contain any elements which are not elements of $\{5, 8\}$?

Since each element of a given set is also an element of the given set, it follows that any set is a subset of itself. That is, if A denotes any set, then $A \subseteq A$.

A set with n elements has 2^n subsets. So, $\{3, 5, 7\}$ has 2^3 , or 8, subsets. They are $\{\}$, $\{3\}$, $\{5\}$, $\{7\}$, $\{3, 5\}$, $\{3, 7\}$, $\{5, 7\}$, and $\{3, 5, 7\}$.

Proper subset of a set: A set A is a *proper subset* of a set B if everything that is a member of A is also a member of B and there is at least one element in B which is not in A. The symbol $\subset$ is used to write $A \subset B$ which is read *A is a proper subset of B*.

Let S denote $\{4, 8\}$. Then S has four subsets, $\{\}$, $\{4\}$, $\{8\}$, and $\{4, 8\}$, but $\{\}$, $\{4\}$, and $\{8\}$ are proper subsets of S while $\{4, 8\}$ is not. CAUTION: Some authors use $\subset$ to mean "is a subset of."

The universal set: Quite often we say we are working only with counting numbers or only with whole numbers or only with integers or, and so on. The set of numbers for any particular discussion is called the *universal set* or *universe* for the discussion. Usually, the letter U is used to denote the universal set. For example, if the discussion is limited to counting numbers then $U = \{1, 2, 3, \dots\}$.

The complement of a set: The *complement* of a set A is the set of all things that are members of the universal set but are not members of set A .

Example 1. If $U = \{2, 3, 4, 5, 6\}$, then the complement of $\{2, 4\}$ is $\{3, 5, 6\}$.

Example 2. If U is the set of counting numbers, then the complement of the set of odd counting numbers is the set of even counting numbers.

It follows from the definition that the complement of $\{ \}$ is U and the complement of U is $\{ \}$.

Chapter I

Exercises

1. List within braces the members of $\{0, 2, 3, 7, 8, 11\}$ which are
 - a. even numbers.
 - b. odd numbers.
 - c. greater than 7.
 - d. less than 2.
2. The set of letters of the English alphabet has how many members?
3. The set for Ex. 2 is which, finite or infinite?
4. True or False?
 - a. $5 \in \{2, 3, 6\}$
 - b. $7 \notin \{2, 3, 6\}$
 - c. $\{0, 1, 2, 3, \dots, 12\}$ has 12 members.
5. Is it true that $\{2, 3, 7\} = \{7, 2, 3\}$?
6. Pair the members of $\{2, 4, 5\}$ with the members of $\{3, 6, 7\}$ in six different ways, each way showing that a one-to-one correspondence exists between $\{2, 4, 5\}$ and $\{3, 6, 7\}$.
7. To which set is $\{0, 4\}$ equivalent, $\{2, 3\}$ or $\{0, 4, 8\}$?
8. Consider the sets named at the right.

List within braces the members of

 - a. $A \cup B$
 - b. $A \cup C$
 - c. $A \cap B$
 - d. $A \cap C$

$A = \{3, 10, 20\}$
 $B = \{3, 9, 10, 15\}$
 $C = \{4, 8, 12\}$

 - e. $A \cup (B \cap A)$
 - f. $A \cap (B \cap A)$
 - g. $\{ \} \cup (A \cap B)$
 - h. $(A \cup B) \cap \{ \}$
9. Let S denote $\{0, 3, 7\}$ and let T denote $\{1, 5, 10\}$. Copy and complete by inserting $\subseteq$ or $\not\subseteq$ to make a true sentence.

- a. $\{3\} \_ ? \_ S$
 - b. $\{0\} \_ ? \_ T$
 - c. $\{1\} \_ ? \_ S$
 - d. $\{0, 7\} \_ ? \_ S$
 - e. $\{ \} \_ ? \_ S$
 - f. $S \_ ? \_ T$

10. A set with 5 members has how many subsets?
11. Write all the 2-element subsets of $\{10, 20, 30\}$.
12. True or False.
 - a. If $A \subseteq B$, then $A \cup B = B$
 - b. If $A \subseteq B$, then $A \cap B = A$
13. Suppose $U = \{1, 2, 3 \dots, 10\}$ and $A = \{3, 4, 5\}$. List within braces the members of the complement of A.
14. Suppose sets A and B are disjoint and set A has 5 members and $A \cup B$ has 7 members. Then set B has how many members?
15. Suppose set S has 7 members and set T has 8 members and $S \cap T$ has 3 members. Then $S \cup T$ has how many members?

Chapter I

Answers for exercises

1. a. $\{0, 2\}$
b. $\{3, 7, 9, 11\}$
c. $\{9, 11\}$
d. $\{0\}$
2. 26
3. Finite
4. a. False
b. True
c. False (It has 13 members)
5. Yes
6. $\begin{array}{cccccc} \{2, 4, 5\} & \{2, 4, 5\} & \{2, 4, 5\} & \{2, 4, 5\} & \{2, 4, 5\} & \{2, 4, 5\} \\ \downarrow \downarrow \downarrow & \swarrow \downarrow \swarrow & \swarrow \downarrow \swarrow & \downarrow \downarrow \downarrow & \swarrow \downarrow \swarrow & \swarrow \downarrow \swarrow \\ \{3, 6, 7\} & \{3, 6, 7\} & \{3, 6, 7\} & \{3, 6, 7\} & \{3, 6, 7\} & \{3, 6, 7\} \end{array}$
7. $\{2, 3\}$
8. a. $\{3, 9, 10, 15, 20\}$
b. $\{3, 4, 8, 10, 12, 20\}$
c. $\{3, 10\}$
d. $\{ \}$
e. $\{3, 9, 10, 15, 20\}$
f. $\{ \}$
g. $\{3, 10\}$
h. $\{ \}$
9. a. $\{3\} \subseteq S$
b. $\{0\} \not\subseteq T$
c. $\{1\} \not\subseteq S$
d. $\{0, 5\} \subseteq S$
e. $\{ \} \subseteq S$
f. $S \not\subseteq T$
10. 32 (or 2^5)
11. $\{10, 20\}, \{10, 30\}, \{20, 30\}$
12. a. True
b. True
13. $\{1, 2, 6, 7, 8, 9, 10\}$
14. 2
15. 12

Chapter II

Exponents

PART I: Help to You in Learning Mathematics

Numbers and numerals: A *number* is an idea --- a concept --- a property. We cannot see a number. A *numeral* is a name for a number. Some numbers have many names. For example, 5, $2 + 3$, and $15 \div 3$ are three numerals for the same number, the number five. The numeral 5 is the *standard numeral* for the number five. Some other types of numerals are: *expanded forms* (e.g., $(2 \times 10) + (3 \times 1)$ is an expanded form for 23); *decimals* (e.g., 1.26); *fractions* (e.g., $\frac{2}{3}$, $\frac{0}{1}$, $\frac{10}{12}$, $\frac{7}{4}$); *mixed forms* (e.g., $2\frac{3}{4}$); *exponent forms* (e.g., 2^3 , 5^2).

Standard numerals: Standard numerals are composed of some or all of the basic symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. These basic symbols are called *digits*. They are the standard numerals for the numbers zero, one, two, three, four, five, six, seven, eight, and nine, respectively. Other standard numerals (e.g., 743) are formed by using the digits and the *place-value principle*.

Although each digit in a standard numeral names a number (the "face value" of the digit in traditional terminology), the digit *in its place* indicates or suggests a number which is the product of the number named by the digit and the value of the place. This is characteristic of a *place-value system of numeration*. In 743, the 4 in its place indicates the number 4×10 , or 40.

Expanded form: In 743, the 7 is in hundred's place, the 4 is in ten's place, and the 3 is in one's place. The number indicated by the 7 in its place is 7×100 , by the 4 in its place is 4×10 , and by the 3 in its place is 3×1 . So, the number 743 may be expressed in *expanded form* as

$$(7 \times 100) + (4 \times 10) + (3 \times 1).$$

Exponent forms: Numerals such as 2^3 , 5^4 , and 10^2 are called *exponent forms*. In an exponent form, the small raised numeral names the *exponent*. The large numeral names the *base*. For 2^3 , the exponent is 3 and the base is 2. The numeral 2^3 is read *two cubed* or *two to the third power*.

If the exponent is a counting number, it indicates how many times the base is used as a factor.

Thus,

$$2^3 = 2 \times 2 \times 2 = 8,$$

$$5^4 = 5 \times 5 \times 5 \times 5 = 625,$$

and $10^2 = 10 \times 10 = 100$.

The numbers 2 , 2^2 , 2^3 , and so on are powers of 2 . The exponent is not named when writing the exponent form for *two to the first power*. The numbers 10 , 10^2 , 10^3 , and so on are powers of 10 . In general, if n represents a counting number, the number named by a^n is the n th power of a .

Definition: $a^0 = 1$ if $a \neq 0$. Examples: $2^0 = 1$; $3^0 = 1$; $10^0 = 1$. So, the place values 1 , 10 , 100 , $1,000$, and so on can now be expressed in exponent form as powers of 10 , that is, as 10^0 , 10^1 , 10^2 , 10^3 , and so on. Because the place values are powers of 10 , our numeration system is a *base-ten* numeration system. In a base-ten system the place values are powers of 10 and the value of a place is ten times the value of the place to the right.

Numbers named by standard numerals may now be expressed in expanded form using exponent forms. For example, the number 743 may be expressed as

$$(7 \times 10^2) + (4 \times 10) + (3 \times 10^0).$$

Also, $258 = 200 + 50 + 8$

$$= (2 \times 100) + (5 \times 10) + (8 \times 1)$$

$$= (2 \times (10 \times 10)) + (5 \times 10) + (8 \times 1)$$

$$= (2 \times 10^2) + (5 \times 10) + (8 \times 10^0).$$

Multiplying with exponent forms: To find $2^3 \times 2^4$, think:

7 factors

$$2^3 \times 2^4 \text{ means } \underbrace{2 \times 2 \times 2}_{3 \text{ factors}} \times \underbrace{2 \times 2 \times 2 \times 2}_{4 \text{ factors}}, \text{ or } 2^7.$$

3 factors 4 factors

So, an exponent form for $2^3 \times 2^4$ is 2^7 . In general, if m and n represent counting numbers, then

$$a^m \times a^n = a^{m+n}.$$

Examples: $5^3 \times 5^2 = 5^{3+2} = 5^5$

$$3^5 \times 3^4 = 3^{5+4} = 3^9$$

$$2 \times 2^5 = 2^{1+5} = 2^6$$

To find $2^5 \times 2^0$, think $2^0 = 1$ so $2^5 \times 2^0 = 2^5 \times 1 = 2^5$.

Dividing with exponent forms: To find $3^5 \div 3^2$, think:

$$\begin{aligned} 3^5 \div 3^2 &= \frac{3^5}{3^2} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = \frac{(3 \times 3) \times (3 \times 3 \times 3)}{(3 \times 3) \times 1} \\ &= \frac{3 \times 3 \times 3}{1}, \text{ or } 3^3. \end{aligned}$$

If m and n represent counting numbers and $m < n$, then

$$a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}.$$

Examples: $2^7 \div 2^5 = 2^{7-5} = 2^2$

$$5^8 \div 5^4 = 5^{8-4} = 5^4$$

$$7^2 \div 7^2 = 7^{2-2} = 7^0$$

To find $2^3 \div 2^5$, think:

$$\begin{aligned} 2^3 \div 2^5 &= \frac{2^3}{2^5} = \frac{2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{(2 \times 2 \times 2) \times 1}{(2 \times 2 \times 2) \times (2 \times 2)} \\ &= \frac{1}{2 \times 2}, \text{ or } \frac{1}{2^2}. \end{aligned}$$

If m and n represent counting numbers and $m < n$, then

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$

Examples: $3^4 \div 3^6 = \frac{1}{3^{6-4}} = \frac{1}{3^2} = \frac{1}{3^2}$

$$5^3 \div 5^7 = \frac{1}{5^{7-3}} = \frac{1}{5^4}$$

In summary,

$$a^m \div a^n = \frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n. \\ 1 & \text{if } m = n. \\ \frac{1}{a^{n-m}} & \text{if } m < n. \end{cases}$$

The notation a^{-n} means $\frac{1}{a^n}$. For example, $2^{-3} = \frac{1}{2^3}$.

Thus,

$$10^{-1} = \frac{1}{10^1} = \frac{1}{10}$$

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$$

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$$

.

.

.

and so on.

Scientific notation: Scientific notation is a special form for naming a positive number. The number is expressed as a product of a number that is greater than or equal to 1 but less than 10 and a power of 10.

Examples:

<u>Number</u>	<u>Scientific Notation</u>
123	1.23×10^2
62	6.2×10
0.85	8.5×10^{-1}
0.0047	4.7×10^{-3}
87,500,000	8.75×10^7
0.000003604	3.604×10^{-6}

Chapter II

Exercises

1. Write the standard numeral for
 - a. 2×52
 - b. $(4 \times 62) + 87$
2. In 2,624 the digit 6 is in which place?
3. In 2,624 the digit 6 in its place indicates what number?
4. Write an exponent form for
 - a. two squared.
 - b. three to the fourth power.
 - c. twenty-five cubed.
 - d. thirty-two.
 - e. sixteen.
5. Write the standard numeral for the number.

a. 2^6	d. $2^2 + 3^4$
b. 5^3	e. $4^0 + 5^2 + 10^3$
c. 10^5	f. $25^2 + 2^3$
6. Express the product with an exponent form.

a. $10^4 \times 10^2$	d. $11^1 \times 11^1$
b. $6^3 \times 6^7$	e. 7×7
c. $8^5 \times 8^0$	f. $8^7 \times 8^2$
7. Express the unknown factor with an exponent form.

a. $4^4 + 4^2$	d. $9^6 + 9^0$
b. $12^8 + 12^2$	e. $5^4 + 5$
c. $10^3 + 10^3$	f. $10^8 + 10^4$
8. Using exponent forms for the powers of 10, write an expanded form for the number.
 - a. 3,954
 - b. 481
 - c. 8,600

9. Express in exponent form:
- | | |
|---------------|-------------------|
| a. 10,000,000 | b. 10,000,000,000 |
|---------------|-------------------|
10. Write the standard numeral for *seven billion six hundred thirty million twenty-seven thousand eighty-eight*.
11. Rename, using scientific notation:
- | | |
|----------------|-----------------|
| a. 3,400 | d. 0.4325 |
| b. 878,000,000 | e. 0.0059 |
| c. 53.62 | f. 0.0000000778 |

Chapter II

Answers for exercises

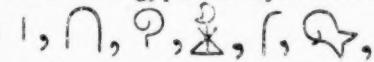
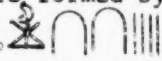
1. a. 104
b. 335
2. in hundred's place (or the third place)
3. 600
4. a. 2^2
b. 3^4
c. 25^3
d. 2^5
e. 4^2 (or 2^4)
5. a. 64
b. 125
c. 100,000
d. 85
e. 1,026
f. 633
6. a. 10^6
b. 6^{10}
c. 8^5 (or 2^{15})
d. 11^2
e. 7^2
f. 8^9 (or 2^{27})
7. a. 4^2
b. 12^6
c. 10^0
d. 9^6
e. 5^3
f. 10^4
8. a. $(3 \times 10^3) + (9 \times 10^2) + (5 \times 10) + (4 \times 10^0)$
b. $(4 \times 10^2) + (8 \times 10) + (1 \times 10^0)$
c. $(8 \times 10^3) + (6 \times 10^2) + (0 \times 10) + (0 \times 10^0)$
9. a. 10^7
b. 10^{10}
10. 7,630,027,088
11. a. 3.4×10^3
b. 8.78×10^8
c. 5.362×10
d. 4.325×10^{-1}
e. 5.9×10^{-3}
f. 7.78×10^{-8}

Chapter III

The Hindu-Arabic System

PART I: Help to You in Learning Mathematics

System of numeration: A system of numeration or numeration system consists of a set of basic symbols and a set of rules with which those symbols may be "combined" to form numerals for a universal set of numbers. For many numeration systems, the basic symbols are referred to as *digits*.

An ancient system of numeration: For the ancient Egyptian system of numeration, the "digits" were the basic symbols , and  which were pictures of a staff, a heelbone, a scroll, a lotus flower, a bent finger, a fish and a man showing astonishment. These basic symbols were numerals for the numbers 1, 10, 100, 1,000, 10,000, 100,000, and 1,000,000 respectively. Other numerals were formed by using the basic symbols and an additive principle. For example,  named the number $1,000 + 10 + 10 + 1 + 1 + 1 + 1$, or 1,024. The order in which the digits appeared was irrelevant. So,  named $100 + 10$, or 110 and  named $10 + 100$, or 110. The digits were not always written horizontally. For example,  named the number 22.

Roman numerals: Numerals in the Roman system of numeration are formed by using the basic symbols I, V, X, L, C, D, and M. These are numerals for 1, 5, 10, 50, 100, 500, and 1,000 respectively. An additive principle is used in forming the numeral VI (VI names the number $5 + 1$, or 6) and a subtractive principle is used in forming the numeral IV (IV names the number $5 - 1$, or 4). Thus the Roman system is both additive and subtractive (the Egyptian system was additive only). If the six numerals IV, IX, XL, XC, CD, and CM (names for 4, 9, 40, 90, 400, and 900 respectively) are treated as single symbols then the number named by a Roman numeral may be found by adding the numbers named by the symbols in the numeral.

Examples:

XIX names $10 + 9$, or 19

CDXIV names $400 + 10 + 4$, or 414

A multiplicative principle is often used for naming greater numbers. A numeral with a bar drawn above it names the number which is 1,000 times the number named by the original numeral. For example $\overline{\text{LV}}$ names $1,000 \times 55$, or 55,000.

The Hindu-Arabic system of numeration: Our system of numeration is the Hindu-Arabic system of numeration and uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 and the place-value principle. The values of the places are powers of 10, so the system is a *decimal* or *base-ten* system of numeration. Also, the value of a place is ten times the value of the place to the right.

For a *standard numeral* such as 743 or a *decimal* such as 24.37, each digit in its place indicates a number which is the product of the number named by the digit and the value of the place.

Example 1. For 743,

the 7 indicates 7×100 , or 700,

the 4 indicates 4×10 , or 40,

and the 3 indicates 3×1 , or 3.

Thus, $743 = 700 + 40 + 3$.

Example 2. For 24.37,

the 2 indicates 2×10 , or 20,

the 4 indicates 4×1 , or 4,

the 3 indicates $3 \times \frac{1}{10}$, or $\frac{3}{10}$ (or 0.3),

and the 7 indicates $7 \times \frac{1}{100}$, or $\frac{7}{100}$ (or 0.07).

Thus, $24.37 = 20 + 4 + \frac{3}{10} + \frac{7}{100} = 20 + 4 + 0.3 + 0.07$.

In expanded form, using exponent forms for the powers of 10, we have

$$743 = (7 \times 10^2) + (4 \times 10) + (3 \times 10^0),$$

$$\text{and } 24.37 = (2 \times 10) + (4 \times 10^0) + (3 \times 10^{-1}) + (7 \times 10^{-2}).$$

Numbers: The set of counting numbers (that is, $\{1, 2, 3, \dots\}$) is the same regardless of which system of numeration we employ in writing names for numbers. The numerals $\overline{\text{I}}$ (Egyptian), X (Roman), and 10 (Hindu-Arabic) each name the same number. By *our number system*, we usually mean the system of *real numbers*. Some real numbers are 0, -5, $\frac{1}{2}$, $\sqrt{2}$, 2.37, and π .

Some of the subsets of the set of real numbers are the set of *counting numbers*, the set of *whole numbers*, the set of *integers*, the set of *rational numbers*, and the set of *irrational numbers*. We shall study these sets in detail in later chapters.

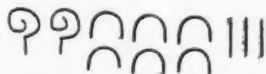
Chapter III

Exercises

1. Write the standard numeral for the number named by the Egyptian numeral.

a. 

b. 

c. 

2. Write an Egyptian numeral for the number.

a. 12

b. 331

c. 47

d. 10,532

3. Was the ancient Egyptian system a place-value system?

4. Write the standard numeral for the number named by the Roman numeral.

a. VIII

d. MMDXC

b. XXXIV

e. XIX

c. CMXLIX

f. LVIII

5. Write a Roman numeral for the number.

a. 24

d. 555

b. 59

e. 2,040

c. 268

f. 42,000

6. Write the standard numeral or a decimal for the number indicated by the 7 in its place.

a. 37,462

b. 2.0735

7. In a base-ten system, the value of a place is how many times the value of the place to the right?

8. Using exponent forms for the powers of 10, write an expanded form for the number.

a. 3.86

b. 0.578

Chapter III

Answers for exercises.

1. a. 2,042
b. 1,020,012
c. 263

2. Sample answers:

a. $\cap ||$ b. $\varphi \varphi \varphi \cap \cap \cap \cap$ c. $\cap \cap \cap \cap |||||$ d. $\varphi \varphi \varphi \varphi \varphi \cap \cap \cap ||$

3. No

4. a. 8
b. 34
c. 949

- d. 2,590
e. 19,000
f. 58,000

5. a. XXIV

b. LIX

c. CCLXVIII

d. DLV

e. MMXL (or $\overline{\text{IIXL}}$)f. $\overline{\text{XLII}}$

6. a. 7,000

b. 0.07

7. Ten

8. a. $(3 \times 10^0) + (8 \times 10^{-1}) + (6 \times 10^{-2})$
b. $(5 \times 10^{-1}) + (7 \times 10^{-2}) + (8 \times 10^{-3})$

Chapter IV

Other Numeration Systems

PART I: Help to You in Learning Mathematics

Base-five system of numeration: There are fourteen dots in Box I. In Box II the fourteen dots are shown in one group of ten with four extra. When we write 14 we mean 1 ten + 4 ones. The numeral 14 is a base-ten numeral. Now let us write a *base-five* numeral for the number of dots. In Box III the dots are shown in two groups of five with four extra. There are 2 fives + 4 ones, so we may write the base-five numeral 24 to express the number of dots. To show that 24 is a base-five numeral, we write 24_{five} (or 24(five)) read *two-four, base five*.

In a *base-five* system of numeration the place values are powers of 5 and the value of a place is five times the value of the place to the right. As with standard numerals in the decimal system, each digit in a base-five numeral indicates a number which is the product of the number named by the digit and the value of the place. For example, in 32_{five} the 3 is in five's place and indicates 3×5 , or 15, while the 2 is in one's place and indicates the number 2×1 , or 2. Thus, 32_{five} means 3 fives + 2 ones, or 17. We write $32_{\text{five}} = 17$ to show that 32_{five} and 17 are two names for the same number. In other words, the base-five numeral 32 names the same number as the base-ten numeral 17.

Thus,

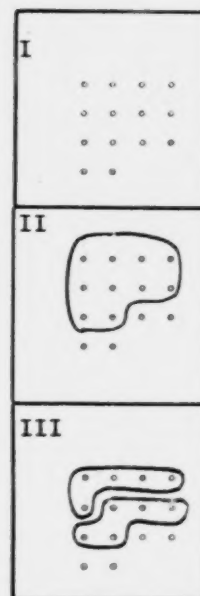
$$32_{\text{five}} = (3 \times 5) + (2 \times 1) = 15 + 2 = 17,$$

$$14_{\text{five}} = (1 \times 5) + (4 \times 1) = 5 + 4 = 9,$$

$$\text{and } 43_{\text{five}} = (4 \times 5) + (3 \times 1) = 20 + 3 = 23.$$

Consider 243_{five} . The 3 is in one's place and indicates 3×1 , or 3, the 4 is in five's place and indicates 4×5 , or 20, and the 2 is in twenty-five's place and indicates 2×25 , or 50. So,

$$243_{\text{five}} = (2 \times 25) + (4 \times 5) + (3 \times 1) = 50 + 20 + 3 = 73.$$



The value of the fourth place is 5^3 , or 125, of the fifth place is 5^4 , or 625, and so on. Therefore,

$$\begin{aligned} 2142_{\text{five}} &= (2 \times 125) + (1 \times 25) + (4 \times 5) + (2 \times 1) \\ &= 250 + 25 + 20 + 2 \\ &= 297, \end{aligned}$$

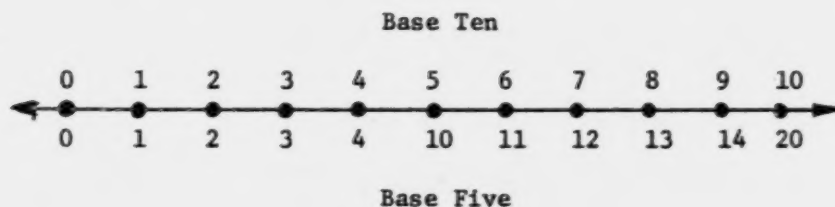
$$\begin{aligned} \text{and } 30403_{\text{five}} &= (3 \times 625) + (0 \times 125) + (4 \times 25) + (0 \times 5) + (3 \times 1) \\ &= 1875 + 0 + 100 + 0 + 3 \\ &= 1978. \end{aligned}$$

In finding a base-ten numeral for a number that is expressed in base five, it may help to use a place-value chart. The place values for base five are, 5^0 , 5^1 , 5^2 , 5^3 , 5^4 , ... or 1, 5, 25, 125, 625, ... Thus, we can chart 2103_{five} as follows:

...	625	125	25	5	1
		2	1	0	3

$$\begin{aligned} \text{Therefore, } 2103_{\text{five}} &= (2 \times 125) + (1 \times 25) + (0 \times 5) + (3 \times 1) \\ &= 250 + 25 + 0 + 3 \\ &= 278. \end{aligned}$$

Renaming with base-five numerals: The whole numbers from zero through ten may be expressed in base five as 0_{five} , 1_{five} , 2_{five} , 3_{five} , 4_{five} , 10_{five} , 11_{five} , 12_{five} , 13_{five} , 14_{five} , and 20_{five} as shown in the following number-line picture.



There are only five digits (basic symbols) needed in a base-five system of numeration. We have been using 0, 1, 2, 3, and 4. To see why the symbol 5 is not needed, consider writing a base-five numeral for the number *twenty-six* --- instead of using "5" and writing 51_{five} , we would write 101_{five} .

Consider the number whose base-ten numeral is 63. To express this number is base five, think: *The greatest power of five which is less than or equal to 63 is 5^2 , or 25. $63 = (2 \times 25) + (2 \times 5) + (3 \times 1)$, so a base-five numeral for 63 is 223_{five} .*

Some other examples:

$$17 = (3 \times 5) + (2 \times 1)$$

$$= 32_{\text{five}}$$

$$29 = (1 \times 25) + (0 \times 5) + (4 \times 1)$$

$$= 104_{\text{five}}$$

$$125 = (1 \times 125) + (0 \times 25) + (0 \times 5) + (0 \times 1)$$

$$= 1000_{\text{five}}$$

Working with other bases: In a base-eight system, the place values are powers of 8 and the value of a place is eight times the value of the place to the right. Thus, 263_{eight} (read *two-six-three, base eight*) means

$$(2 \times 8^2) + (6 \times 8^1) + (3 \times 8^0)$$

$$\text{or } (2 \times 64) + (6 \times 8) + (3 \times 1)$$

$$\text{or } 128 + 48 + 3$$

$$\text{or } 179.$$

To express a number in base eight, the symbols 0, 1, 2, 3, 4, 5, 6, and 7 are used for the digits of the system. Examples of renaming with base-eight numerals:

$$22 = (2 \times 8) + (6 \times 1)$$

$$= 26_{\text{eight}}$$

$$67 = (1 \times 64) + (0 \times 8) + (3 \times 1)$$

$$= 103_{\text{eight}}$$

$$223 = (3 \times 64) + (3 \times 8) + (7 \times 1)$$

$$= 337_{\text{eight}}$$

A *base-two* (or *binary*) system of numeration has place values which are powers of 2. The system requires only two digits, 0 and 1. Consider 10111_{two} . It may be charted as follows:

...	32	16	8	4	2	1
		1	0	1	1	1

$$\begin{aligned}
 \text{Thus, } 10111_{\text{two}} &= (1 \times 16) + (0 \times 8) + (1 \times 4) + (1 \times 2) + (1 \times 1) \\
 &= 16 + 0 + 4 + 2 + 1 \\
 &= 23.
 \end{aligned}$$

That is, the base-two numeral 10111 and the base-ten numeral 23 are names for the same number.

Examples of renaming with base-two numerals:

$$\begin{aligned}
 9 &= (1 \times 8) + (0 \times 4) + (0 \times 2) + (1 \times 1) \\
 &= 1001_{\text{two}}
 \end{aligned}$$

$$\begin{aligned}
 45 &= (1 \times 32) + (0 \times 16) + (1 \times 8) + (1 \times 4) + (0 \times 2) + (1 \times 1) \\
 &= 101101_{\text{two}}
 \end{aligned}$$

A base-two system requires two digits, a base-three system requires three digits, and so on. So, a *base-twelve system* requires twelve digits. Consider a base-twelve system which uses the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, and E for digits. (T represents the number ten and E represents the number eleven.) In $3T_{\text{twelve}}$, the 3 is in twelve's place and the T is in one's place. Therefore,

$$\begin{aligned}
 3T_{\text{twelve}} &= (3 \times 12) + (10 \times 1) \\
 &= 36 + 10 \\
 &= 46.
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } 2E4_{\text{twelve}} &= (2 \times 144) + (11 \times 12) + (4 \times 1) \\
 &= 288 + 132 + 4 \\
 &= 424.
 \end{aligned}$$

Examples of renaming with base-twelve numerals:

$$\begin{aligned}
 18 &= (1 \times 12) + (6 \times 1) \\
 &= 16_{\text{twelve}}
 \end{aligned}$$

$$\begin{aligned}
 23 &= (1 \times 12) + (11 \times 1) \\
 &= 1E_{\text{twelve}}
 \end{aligned}$$

$$\begin{aligned}
 124 &= (10 \times 12) + (4 \times 1) \\
 &= T4_{\text{twelve}}
 \end{aligned}$$

Arithmetic in other bases: The basic addition facts and basic multiplication facts for base five are shown in the tables below. Use the tables to verify that $1_{\text{five}} + 4_{\text{five}} = 10_{\text{five}}$ and that $3_{\text{five}} \times 4_{\text{five}} = 22_{\text{five}}$.

+	0	1	2	3	4	×	0	1	2	3	4
0	0	1	2	3	4	0	0	0	0	0	0
1	1	2	3	4	10	1	0	1	2	3	4
2	2	3	4	10	11	2	0	2	4	11	13
3	3	4	10	11	12	3	0	3	11	14	22
4	4	10	11	12	13	4	0	4	13	22	31

Using the basic addition facts for base five we can add with base-five numerals. For example, study the following work for finding $14_{\text{five}} + 23_{\text{five}}$. (In the vertical form for addition we shall use 14 (five) for 14_{five} , and so on.)

Add ones: $3 + 4 = 12$ (remember -- we are working in base five). Rename 12 ones as 1 five + 2 ones. Write a 2 in one's place in the numeral for the sum and write a 1 in the fives column

$$\begin{array}{r} 1 \\ 14(\text{five}) \\ + 23(\text{five}) \\ \hline 42(\text{five}) \end{array}$$

Add fives: $2 + 1 + 1$ (carry) = 4. Write a 4 in five's place in the numeral for the sum. Therefore,

$$14_{\text{five}} + 23_{\text{five}} = 42_{\text{five}}$$

We may check our work by renaming with base-ten numerals and adding in base ten.

$$\begin{array}{rcl} 14(\text{five}) & \longrightarrow & 9 \\ + 23(\text{five}) & \longrightarrow & + 13 \\ \hline 42(\text{five}) & \longrightarrow & 22 \end{array}$$

Now let us use the basic multiplication facts for base five to find $3_{\text{five}} \times 34_{\text{five}}$.

Multiply ones: $3 \times 4 = 22$. Think of 22 ones as 2 fives + 2 ones. Write a 2 in one's place in the numeral for the product. Remember 2 fives.

$$\begin{array}{r} 34(\text{five}) \\ \times 3(\text{five}) \\ \hline 212(\text{five}) \end{array}$$

Multiply fives: $3 \times 3 = 14$. 14 fives + 2 fives = 21 fives. Think of 21 fives as 2 twenty-fives + 1 five. Write a 1 in five's place and a 2 in twenty-five's place in the numeral for the product.

We may check the work as follows:

$$\begin{array}{rcl} 34(\text{five}) & \longrightarrow & 19 \\ \times 3(\text{five}) & \longrightarrow & \times 3 \\ \hline 212(\text{five}) & \longrightarrow & 57 \end{array}$$

Examples of addition and multiplication work with base-two numerals:

$$\begin{array}{r} 1111 \\ 11011(\text{two}) \\ + 1101(\text{two}) \\ \hline 101000(\text{two}) \end{array}$$

$$\begin{array}{r} 1101(\text{two}) \\ \times 11(\text{two}) \\ \hline 1101 \\ 1101 \\ \hline 100111(\text{two}) \end{array}$$

Base Two

+	0	1
0	0	0
1	1	10

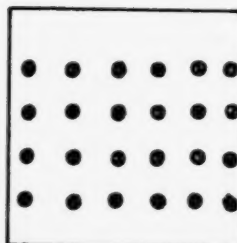
×	0	1
0	0	0
1	0	1

Chapter IV

Exercises

1. Express the number of dots in the box at the right with

- a. a base-five numeral.
b. a base-eight numeral.
c. a base-two numeral.



2. Copy and complete the following table:

Base	Base	Base	Base
<u>Ten</u>	<u>Five</u>	<u>Eight</u>	<u>Twelve</u>
3	<u> ?</u>	<u> ?</u>	<u> ?</u>
10	<u> ?</u>	<u> ?</u>	<u> ?</u>
<u> ?</u>	<u> ?</u>	<u> ?</u>	30
65	<u> ?</u>	<u> ?</u>	<u> ?</u>
<u> ?</u>	<u> ?</u>	<u> ?</u>	E6

3. Do 1100_{two} and 110_{three} name the same number?
4. Do 11000_{two} and 1100_{three} name the same number?
5. How many digits (basic symbols) are needed in
- a. a base-seven system of numeration?
- b. a base-twenty system of numeration?
6. Work the following base-five addition problems. Check your work by renaming in base ten.
- a. Add
- $$\begin{array}{r} 343(\text{five}) \\ + 43(\text{five}) \\ \hline \end{array}$$
- b. Add
- $$\begin{array}{r} 1431(\text{five}) \\ + 431(\text{five}) \\ \hline 234(\text{five}) \end{array}$$

7. Work the following base-five multiplication problems. Check your work by renaming in base ten.
- a. $244(\text{five})$
 $\times 42(\text{five})$
- b. $3404(\text{five})$
 $\times 343(\text{five})$
8. Copy and complete: In a base-six system, the place values are powers of ? and the value of a place is ? times the value of the place to the ? .
9. Show the basic addition facts and the basic multiplication facts for base three.
10. Work the following base-three arithmetic problems.
- a. $212(\text{three})$
 $+ 22(\text{three})$
- b. $1021(\text{three})$
 $\times 202(\text{three})$

Chapter IV

Answers for exercises

1. a. 44_{five} (or 44(five) or simply 44)
 b. 30_{eight} (or 30 (eight) or simply 30)
 c. 11000_{two} (or 11000(two) or simply 11000)

2. Base	Base	Base	Base
<u>Ten</u>	<u>Five</u>	<u>Eight</u>	<u>Twelve</u>
3	3	3	3
10	20	12	T
36	121	44	30
65	230	101	55
138	1023	212	E6

3. Yes

4. No

5. a. Seven

b. Twenty

6. a. $\begin{array}{r} 11 \\ 343(\text{five}) \longrightarrow 98 \\ \underline{43(\text{five})} \longrightarrow 23 \\ 441(\text{five}) \longrightarrow 121 \end{array}$ b. $\begin{array}{r} 221 \\ 1431(\text{five}) \longrightarrow 241 \\ \underline{431(\text{five})} \longrightarrow 116 \\ \underline{233(\text{five})} \longrightarrow 68 \\ 3200(\text{five}) \longrightarrow 425 \end{array}$
7. a. $\begin{array}{r} 244(\text{five}) \longrightarrow 74 \\ \times 42(\text{five}) \longrightarrow \times 22 \\ 1043 \qquad \qquad 148 \\ \underline{2141} \qquad \qquad \underline{148} \\ 23003(\text{five}) \longrightarrow 1628 \end{array}$ b. $\begin{array}{r} 3404(\text{five}) \longrightarrow 479 \\ \times 343 \longrightarrow \times 98 \\ 21222 \qquad \qquad 3832 \\ 30131 \qquad \qquad \underline{4311} \\ \underline{21222} \qquad \qquad 46942 \\ 3000232(\text{five}) \longrightarrow \end{array}$

8. In a base-six system, the place values are powers of 6 and the value of a place is six times the value of the place to the right.

9.

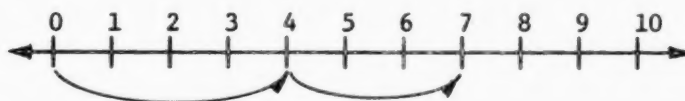
+	0	1	2
0	0	1	2
1	1	2	10
2	2	10	11

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

10. a. 1011(three)
 b. 221012(three)

PART I: Help to You in Learning Mathematics

Addition as a binary operation: A *binary operation* $*$ defined on a set S is a rule which assigns to each pair a and b of S , in that order, an element denoted by $a * b$. For example, addition is a binary operation on the set of whole numbers --- to each pair of whole numbers a and b , there is assigned an element denoted by $a + b$. More specifically, to 2 and 3 is assigned $2 + 3$, or 5, to 3 and 2 is assigned $3 + 2$, or 5, to 4 and 4 is assigned $4 + 4$, or 8, and so on. In more traditional terminology, addition is an operation on two numbers called *addends* resulting in a unique number called the *sum*. Thus, for $4 + 3 = 7$, 4 and 3 are the addends and 7 is the sum. When the addends are known, we *add* to find the sum. The number-line picture below suggests that $4 + 3 = 7$.



Using sets to explain addition: Addition on whole numbers may be explained by using disjoint finite sets. For example, $\{2, 5\}$ and $\{4, 7, 8\}$ are disjoint finite sets. Let $n\{2, 5\}$ represent the number of elements in $\{2, 5\}$ and let $n\{4, 7, 8\}$ represent the number of elements in $\{4, 7, 8\}$. Thus, $n\{2, 5\} = 2$ and $n\{4, 7, 8\} = 3$. Now,

$$\{2, 5\} \cup \{4, 7, 8\} = \{2, 4, 5, 7, 8\}$$

$$\text{and } n\{2, 5\} + n\{4, 7, 8\} = n\{2, 4, 5, 7, 8\}.$$

Then is, $2 + 3 = 5$.

If A and B are disjoint finite sets then the number of elements in A plus the number of elements in B is equal to the number of elements in $A \cup B$. This is true even if one or both of A and B is the empty set (the number of elements in the empty set is 0).

$$\{ \} \cup \{2, 4, 6\} = \{2, 4, 6\}$$

$$n\{ \} + n\{2, 4, 6\} = n\{2, 4, 6\}$$

$$0 + 3 = 3$$

Consider what happens when the finite sets are not disjoint. For example,

$$\{2, 3\} \cup \{3, 5, 6\} = \{2, 3, 5, 6\}$$

$$\text{but } n\{2, 3\} + n\{3, 5, 6\} \neq n\{2, 3, 5, 6\}.$$

That is, $2 + 3 \neq 4$.

The concept of closure: Let $*$ be a binary operation defined on a set S . If, for each pair a and b of S , $a * b$ is an element of S , then the set S is said to be *closed* with respect to the operation $*$.

The set of counting numbers, $\{1, 2, 3, \dots\}$, is closed with respect to addition. In other words, the sum of any two counting numbers is a counting number.

The set of even counting numbers, $\{2, 4, 6, \dots\}$, is closed with respect to addition (the sum of any two even counting numbers is an even counting number). However, the set of odd counting numbers, $\{1, 3, 5, \dots\}$, is *not* closed with respect to addition (consider: 3 and 7 are odd counting numbers, but the sum $3 + 7$, or 10, is not an odd counting number).

The set of whole numbers, $\{0, 1, 2, 3, \dots\}$, is closed with respect to addition (the sum of any two whole numbers is a whole number).

The Commutative Property of Addition: Let $*$ be a binary operation defined on a set S . If $a * b = b * a$ for all a and b in S , then $*$ is said to be a *commutative operation*. Consider the operation addition as defined on the set of whole numbers. $2 + 3 = 3 + 2$, $4 + 7 = 7 + 4$, $0 + 9 = 9 + 0$, and so on. That is, for any whole numbers a and b , the mathematical sentence $a + b = b + a$ is true. This illustrates the Commutative Property of Addition. In more traditional terminology, the Commutative Property of Addition states that the order of the addends may be changed without changing the sum.

The Associative Property of Addition: Let $*$ be a binary operation defined on a set S . If $(a * b) * c = a * (b * c)$ for all a , b , and c in S , then $*$ is said to be an *associative operation*. Consider the operation addition as defined on the set of whole numbers.

$$(2 + 3) + 5 = 2 + (3 + 5),$$

$$(4 + 0) + 2 = 4 + (0 + 2),$$

$$(3 + 3) + 3 = 3 + (3 + 3),$$

and so on.

That is, for any whole numbers a , b , and c , the mathematical sentence

$$(a + b) + c = a + (b + c)$$

is true. This illustrates the Associative Property of Addition. In more traditional terminology, the Associative Property of Addition states that the grouping of the addends may be changed without changing the sum.

The additive identity: Let $*$ be a binary operation defined on a set S . If S contains an element e such that $e * a = a * e = a$ for all a in S , then e is called the *identity element* for the operation $*$. Consider the operation addition as defined on the set of whole numbers. Since 0 is a whole number and since $0 + a = a + 0 = a$ for any whole number a , 0 is the *identity element for addition* or the *additive identity*.

Casting out nines: "Casting out nines" is one of many methods for checking addition work. The method involves finding the "excess of 9's" for a number. The excess of 9's for a number is the remainder when the number is divided by 9. That is, the excess is the remainder when the greatest whole-number multiple of 9 that is less than or equal to the number is subtracted from the number.

<u>Number</u>		<u>Excess of 9's</u>
816	$816 \div 9 = 90$, remainder 6 (or $816 - (90 \times 9) = 6$)	6

A shorter way to find the excess of 9's for a number is to find the sum of the values of the digits and then find the excess of 9's for that sum (it will be the same as the excess of 9's for the original number). Some examples:

<u>Number</u>	<u>Sum of Values of Digits</u>	<u>Excess of 9's</u>
816	15	6
279	18	0
4057	16	7

To check addition work, find the excess of 9's for each addend, add those excesses and then find the excess of 9's for the sum of the excesses. The result should be the same as the excess of 9's for the sum of the addends. For example,

$$\begin{array}{rcl}
 312 & \rightarrow & 3 + 1 + 2 = 6 \rightarrow 6 \\
 + 556 & \rightarrow & 5 + 5 + 6 = 16 \rightarrow 7 \\
 868 & \rightarrow & 8 + 6 + 8 = 22 \rightarrow 4
 \end{array}
 \quad
 \begin{array}{l}
 6 \rightarrow 6 + 7 = 13 \rightarrow 4 \\
 7 \rightarrow 6 + 7 = 13 \rightarrow 4 \\
 4 \rightarrow 6 + 7 = 13 \rightarrow 4
 \end{array}$$

Also,

$$\begin{array}{rcl}
 312 \div 9 = 34, \text{ remainder } 6 & \nearrow & \\
 + 556 \div 9 = 61, \text{ remainder } 7 & \nearrow & 6 + 7 = 13 \rightarrow 4 \\
 868 \div 9 = 96, \text{ remainder } 4 & \longleftarrow &
 \end{array}$$

If the excess of 9's for the sum of the excesses of 9's for the addends is not the same as the excess of 9's for the sum of the addends, then there is an error in the addition work. However, even if they are the same, there still might be an error. For example, suppose that in adding 312 and 556 the result was found to be 877 (an incorrect result) --- the excess of 9's for the sum of the addends would still be 4 and casting out nines would not indicate that an error had been made.

Casting out elevens: "Casting out elevens" is another method for checking addition work. The "excess of 11's" for a number is the remainder when the number is divided by 11. That is, the excess is the remainder when the greatest whole-number multiple of 11 that is less than or equal to the number is subtracted from the number.

<u>Number</u>		<u>Excess of 11's</u>
8697	$8697 \div 11 = 790, \text{ remainder } 7$ $(8697 - (790 \times 11) = 7)$	7

Another way to find the excess of 11's for a number is:

Step 1: find the sum of the values of the digits in the odd-numbered places (1st place, 3rd place, 5th place, and so on). For 8697 this sum is $7 + 6$, or 13.

Step 2: find the sum of the values of the digits in the even-numbered places (2nd place, 4th place, 6th place, and so on). For 8697 this sum is $9 + 8$, or 17.

Step 3: subtract the sum found in step 2 from the sum found in step 1 (if the sum in step 2 is greater than the sum in step 1, add 11 to the sum in step 1 before subtracting) and then find the excess of 11's in the result. That excess will be the excess of 11's for the original number.

<u>Number</u>	
8697	<i>Step 1:</i> $7 + 6 = 13 \longrightarrow 13 + 11 = 24$ <i>Step 2:</i> $9 + 8 = 17 \longrightarrow 17$ <i>Step 3:</i> 7

To check addition work, find the excess of 11's for each addend, add those excesses and then find the excess of 11's for the sum of the excesses. The result should be the same as the excess of 11's for the sum of the addends.

For example,

$$\begin{array}{l}
 8697 \longrightarrow (7 + 6) - (9 + 8) \longrightarrow (13 + 11) - 17 \longrightarrow 7 \\
 + 4938 \longrightarrow (8 + 9) - (3 + 4) \longrightarrow 17 - 7 = 10 \longrightarrow 10 \\
 13635 \longrightarrow (5 + 6 + 1) - (3 + 3) \longrightarrow 12 - 6 \longrightarrow 6
 \end{array}$$

$7 + 10 = 17 \longrightarrow 6$
 $6 \longleftarrow =$

Also,

$$\begin{array}{l}
 8697 \div 11 = 790, \text{ remainder } 7 \\
 + 4938 \div 11 = 448, \text{ remainder } 10 \\
 13635 \div 11 = 1239, \text{ remainder } 6
 \end{array}$$

$7 + 10 = 17 \longrightarrow 6$
 $6 \longleftarrow =$

As with the method of casting out nines, if the excess of 11's for the sum of the excesses of 11's for the addends is not the same as the excess of 11's for the sum of the addends, then there is an error in the addition work.

Chapter V

Exercises

1. Which does n represent, *an addend* or *the sum*?
 - a. $23 + 5 = n$
 - b. $17 = n + 12$
2. Copy and complete: The mathematical sentence $2 + 30 = 30 + 2$ illustrates the ? Property of Addition.
3. Copy and complete: The mathematical sentence $(2 + 0) + 4 = 2 + (0 + 4)$ illustrates the ? Property of Addition.
4. Suppose a and b represent two whole numbers and $a + b = a$. What number does b represent?
5. Using sets to explain addition involves the union of what kind of sets?
6. Write *Closed* or *Not Closed*.
 - a. $\{5, 10, 15, 20, \dots\}$ with respect to addition
 - b. $\{3, 30, 300, \dots\}$ with respect to addition
 - c. $\{0\}$ with respect to addition
 - d. $\{0, 1\}$ with respect to addition
7. Use the short way to find the excess of 9's for
 - a. 385.
 - b. 28673.
8. Find the excess of 11's for
 - a. 273856.
 - b. 500949.
9. Use "casting out nines" to check the following addition work.

$$\begin{array}{r} 374 \\ 562 \\ \hline 377 \\ 1303 \end{array}$$

10. Use "casting out elevens" to check the following addition work.

2637

4887

5693

13117

Chapter V

Answers for Exercises

1. a. the sum
b. an addend
2. The mathematical sentence $2 + 30 = 30 + 2$ illustrates the Commutative Property of Addition.
3. The mathematical sentence $(2 + 0) + 4 = 2 + (0 + 4)$ illustrates the Associative Property of Addition.
4. Zero
5. disjoint sets (more specifically, disjoint finite sets)
6. a. Closed
b. Not Closed ($3 + 30 = 33$ and $33 \notin \{3, 30, 300, \dots\}$)
c. Closed
d. Not Closed ($1 + 1 = 2$ and $2 \notin \{0, 1\}$)
7. a. 7 ($3 + 8 + 5 = 16 \rightarrow 7$)
b. 8 ($2 + 8 + 6 + 7 + 3 = 26 \rightarrow 8$)
8. a. 0 ($21 - 10 = 11 \rightarrow 0$)
b. 9 ($18 - 9 = 9$)

9.
$$\begin{array}{rcl} 374 & \rightarrow & 3 + 7 + 4 = 14 \rightarrow 5 \\ 562 & \rightarrow & 5 + 6 + 2 = 13 \rightarrow 4 \\ \hline 377 & \rightarrow & 3 + 7 + 7 = 17 \rightarrow 8 \\ 1303 & \rightarrow & 1 + 3 + 0 + 3 = 7 \end{array}$$
- $5 + 4 + 8 = 17 \rightarrow 8$
- not equal - so there is an error in the addition work

10.
$$\begin{array}{rcl} & \text{Excess of 11's} & \\ 2637 & \rightarrow & 8 \\ 4887 & \rightarrow & 3 \\ \hline 5693 & \rightarrow & 6 \\ 13117 & \rightarrow & 5 \end{array}$$
- $8 + 3 + 6 = 17 \rightarrow 6$
- not equal - so there is an error in the addition work

Chapter VI

Subtraction

PART I: Help to You in Learning Mathematics

The addends-sum relationship: When the addends are known, we *add* to find the sum. When the sum and one addend are known, we *subtract* to find the missing addend (or unknown addend).

The four mathematical sentences at the right illustrate the *addends-sum* relationship. For each sentence, 8 is the sum and 5 and 3 are the addends.

$$8 = 5 + 3$$

$$8 = 3 + 5$$

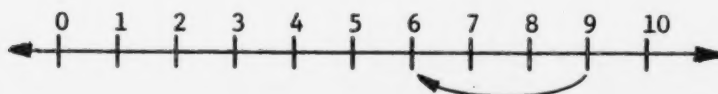
$$8 - 5 = 3$$

$$8 - 3 = 5$$

For $17 - n = 12$, 17 is the sum, 12 is the known addend, and n represents the missing addend. To find the missing addend, we subtract the known addend from the sum. In more traditional terminology, 17 is the *minuend*, 12 is the *subtrahend*, and n represents the *difference*. To find the difference, we subtract the subtrahend from the minuend. Regardless of the terminology used, $n = 17 - 12$, or 5.

Subtraction is sometimes referred to as the operation which "reverses" addition, or the operation which is the "inverse" of addition, or the operation which "undoes" addition.

The number-line picture below suggests that $9 - 3 = 6$.



Using sets to explain subtraction: Let U , the universal set, be $\{2, 3, 7, 8\}$.

Then $n(U) = n\{2, 3, 7, 8\} = 4$.

Let A denote $\{2, 3, 8\}$. Then $A \subseteq U$, that is, A is a subset of U . The complement of A is $\{7\}$. (Recall that the complement of a set A is the set of all things that are members of the universal set but are not members of set A .) Set A and its complement are disjoint finite sets and their union is the universal set.

$$\{2, 3, 8\} \cup \{7\} = \{2, 3, 7, 8\}$$

Thus,

$$n\{2, 3, 8\} + n\{7\} = n\{2, 3, 7, 8\}.$$

That is,

$$3 + 1 = 4.$$

Also, $4 - 3 = 1$. That is, $n\{2, 3, 7, 8\} - n\{2, 3, 8\} = n\{7\}$. In other words, if the universal set is finite, then the number of members in the universal set minus the number of members in a given subset is equal to the number of members in the complement of the given subset.

Subtraction as a binary operation: First, let us review the binary operation addition.

- (1) The set of whole numbers is closed with respect to addition.
- (2) Addition is a commutative operation.
- (3) Addition is an associative operation.
- (4) The set of whole numbers contains an identity element for addition.

Now let us consider subtraction.

- (1) *The set of whole numbers is not closed with respect to subtraction.*

That is, it is not true that for each pair a and b of whole numbers, $a - b$ is a whole number. For example, 3 and 5 are whole numbers but $3 - 5$, or -2 , is not a whole number. In later chapters, we will consider sets which are closed with respect to subtraction. One such set is the set of integers.

- (2) *Subtraction is not a commutative operation.* That is, it is not true that $a - b = b - a$ for all a and b . For example, $5 - 3 \neq 3 - 5$.

- (3) *Subtraction is not an associative operation.* That is, it is not true that $(a - b) - c = a - (b - c)$ for all a , b , and c . For example, $(8 - 6) - 2 \neq 8 - (6 - 2)$.

- (4) *The set of whole numbers does not contain an identity element for subtraction.* That is, the set of whole numbers does not contain an element e such that $e - a = a - e = a$ for each whole number a . Zero is not such an element because although, in general, $a - 0 = a$, it is not true that $0 - a = a - 0 = a$ for all a . (Since it is true that $a - 0 = a$ for all a , the number 0 is a *right-identity* element for subtraction.)

The decomposition method: The *decomposition* method of doing subtraction work involves renaming the sum (minuend). For example, study the work at the right for finding $52 - 17$.

Subtract

$$\begin{array}{r} 4 \quad 12 \\ 52 \\ \underline{17} \\ 35 \end{array}$$

Subtract ones: We cannot subtract 7 ones from 2 ones, so we rename 52 as 4 tens + 12 ones. $12 - 7 = 5$, so we write a 5 in one's place.

Subtract tens: $4 - 1 = 3$, so we write a 3 in ten's place.

The equal-additions method: The *equal-additions* method of doing subtraction work makes use of the principle of compensation---if the same number is added to both the sum (minuend) and the known addend (subtrahend), then the unknown addend (difference) remains unchanged.

$$\text{If } a - b = c, \text{ then } (a + k) - (b + k) = c.$$

The work at the right illustrates the equal-additions method. The number 10 is added to both the sum and the known addend, to the sum in the form of 10 ones and to the known addend in the form of 1 ten. Study the work shown.

Subtract

$$\begin{array}{r} 12 \\ 5 \cancel{2} \\ \underline{2\cancel{1}7} \\ 35 \end{array}$$

Of the two methods, the decomposition method and the equal-additions method, the decomposition method is the more popular method.

The complementary method: The *complementary* method of doing subtraction work was popular in the eighteenth century. It involves finding the complement of a number --- the *complement of a number* is the result of subtracting the number from the least power of 10 which is greater than or equal to the given number. Some examples:

<u>Number</u>	<u>Work</u>	<u>Complement</u>
6	$10 - 6$	4
17	$100 - 17$	83
498	$1000 - 498$	502

To find the unknown addend by the complementary method: From the result of adding the complement of the known addend (subtrahend) and the sum (minuend), we subtract the least power of 10 that is greater than or equal to the known addend. For example, to find $52 - 17$, we find the complement of 17 --- $100 - 17 = 83$, add 83 and 52 --- $83 + 52 = 135$, and then subtract 100 from 135 --- $135 - 100 = 35$. Thus, $52 - 17 = 35$. In general, to find $a - b$, if k is the least power of 10 which is greater than or equal to b , then

$$a - b = (a + (k - b)) - k.$$

Casting out nines: "Casting out nines" is one of many methods for checking subtraction work. (See Chapter 5 if you have forgotten how to find the excess of 9's for a number.)

To check subtraction work, subtract the excess of 9's for the subtrahend from the excess of 9's for the minuend (to avoid negative numbers, 9 may

be added to the excess for the minuend before subtracting). The result should be the same as the excess of 9's for the difference. If it is not, then there is an error in the subtraction work.

Example:

$$\begin{array}{r}
 218 \longrightarrow 2 + 1 + 8 = 11 \longrightarrow 2 \longrightarrow 2 + 9 = 11 \\
 - 179 \longrightarrow 1 + 7 + 9 = 17 \longrightarrow 8 \longrightarrow 8 \\
 \hline
 39 \longrightarrow 3 + 9 = 12 \longrightarrow 3
 \end{array}
 \begin{array}{l}
 \nearrow 11 - 8 = 3 \\
 \longleftarrow = \\
 \nearrow
 \end{array}$$

equal - so the subtraction work is probably correct

Also,

$$\begin{array}{r}
 218 \div 9 = 24, \text{ remainder } 2 \\
 - 179 \div 9 = 19, \text{ remainder } 8 \\
 \hline
 39 \div 9 = 4, \text{ remainder } 3
 \end{array}
 \begin{array}{l}
 \nearrow (2 + 9) - 8 = 3 \\
 \longleftarrow = \\
 \nearrow
 \end{array}$$

Casting out elevens: "Casting out elevens" is another method for checking subtraction work. (See Chapter 5 if you have forgotten how to find the excess of 11's for a number.)

To check subtraction work, subtract the excess of 11's for the subtrahend from the excess of 11's for the minuend (to avoid negative numbers, 11 may be added to the excess for the minuend before subtracting). The result should be the same as the excess of 11's for the difference. If it is not, then there is an error in the subtraction work.

Example:

$$\begin{array}{r}
 218 \longrightarrow (8 + 2) - 1 \longrightarrow 10 - 1 = 9 \\
 - 179 \longrightarrow (9 + 1) - 7 \longrightarrow 10 - 7 = 3 \\
 \hline
 39 \longrightarrow 9 - 3 = 6
 \end{array}
 \begin{array}{l}
 \nearrow 9 - 3 = 6 \\
 \longleftarrow = \\
 \nearrow
 \end{array}$$

equal - so the subtraction work is probably correct

Also,

$$\begin{array}{r}
 218 \div 11 = 19, \text{ remainder } 9 \\
 - 179 \div 11 = 16, \text{ remainder } 3 \\
 \hline
 39 \div 11 = 3, \text{ remainder } 6
 \end{array}
 \begin{array}{l}
 \nearrow 9 - 3 = 6 \\
 \longleftarrow = \\
 \nearrow
 \end{array}$$

Subtraction work with base-five numerals:

The basic addition facts for base five are shown in the table at the right. We may use this table to find basic subtraction facts for base five. Thus,

$$3 - 1 = 2,$$

$$10 - 2 = 3,$$

$$11 - 3 = 3,$$

and so on

(remember --- we are

working in base five).

Now let us find $43_{\text{five}} - 14_{\text{five}}$.

Subtract ones: We cannot subtract 4 ones from 3 ones so we rename 43 as 3 fives + 13 ones. $13 - 4 = 4$, so we write a 4 in one's place.

Subtract fives: $3 - 1 = 2$, so we write a 2 in five's place.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

$$\begin{array}{r}
 3 \quad 13 \\
 \cancel{4}3(\text{five}) \\
 - 14(\text{five}) \\
 \hline
 2 \quad 4(\text{five})
 \end{array}$$

Chapter VI

Exercises

1. Which does n represent, *the unknown addend* or *the sum*? $23 - n = 18$
2. When the sum and one addend are known, what operation is used to find the unknown addend?
3. Copy and complete: The mathematical sentence $3 - 8 \neq 8 - 3$ illustrates that subtraction is not ? .
4. Copy and complete: The mathematical sentence $(9 - 8) - 4 \neq 9 - (8 - 4)$ illustrates that subtraction is not ? .
5. When using the decomposition method in subtraction work, which is true, a, b, or c?
 - a. only the known addend is renamed
 - b. only the sum is renamed
 - c. both the known addend and the sum are renamed
6. Find the complement of the number.
 - a. 18
 - b. 275
 - c. 2,045
7. Use "casting out nines" to check the following subtraction work.

$$\begin{array}{r} 5768 \\ - 1977 \\ \hline 3891 \end{array}$$
8. Use "casting out elevens" to check the following subtraction work.

$$\begin{array}{r} 29476 \\ - 17874 \\ \hline 11802 \end{array}$$
9. Copy and complete:
 - a. $11_{\text{five}} - 2_{\text{five}} = \text{ ? }$
 - b. $11_{\text{eight}} - 2_{\text{eight}} = \text{ ? }$
10. Work the following subtraction problems.
 - a. $3123(\text{five})$
 $- \underline{243}(\text{five})$
 - b. $624(\text{eight})$
 $- \underline{255}(\text{eight})$

Chapter VI

Answers for exercises

1. the unknown addend
2. Subtraction
3. The mathematical sentence $3 - 8 \neq 8 - 3$ illustrates that subtraction is not commutative.
4. The mathematical sentence $(9 - 8) - 4 \neq 9 - (8 - 4)$ illustrates that subtraction is not associative.
5. b
6. a. 82 b. 725 c. 7,955

7.
$$\begin{array}{r} 5768 \rightarrow 5 + 7 + 6 + 8 = 26 \rightarrow 8 \\ - 1977 \rightarrow 1 + 9 + 7 + 7 = 24 \rightarrow 6 \\ 3891 \rightarrow 3 + 8 + 9 + 1 = 21 \rightarrow 3 \end{array}$$

$8 - 6 = 2$

not equal - so there is an error in the subtraction work

8. Excess of 11's

$$\begin{array}{r} 29476 \rightarrow 7 \\ - 17874 \rightarrow 10 \\ 11802 \rightarrow 10 \end{array}$$

$(7 + 11) - 10 = 8$

not equal - so there is an error in the subtraction work

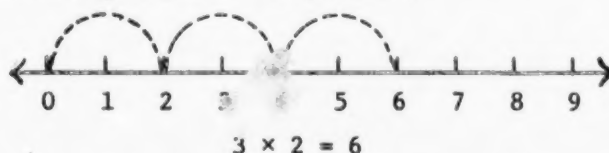
9. a. $11_{\text{five}} - 2_{\text{five}} = 4_{\text{five}}$
b. $11_{\text{eight}} - 2_{\text{eight}} = 7_{\text{eight}}$
10. a.
$$\begin{array}{r} 3 \ 1 \ 2 \ 3(\text{five}) \rightarrow 413 \\ - \ 2 \ 4 \ 3(\text{five}) \rightarrow - \ 73 \\ \hline 2 \ 3 \ 3 \ 0(\text{five}) \rightarrow 340 \end{array}$$

b.
$$\begin{array}{r} 6 \ 2 \ 4(\text{eight}) \rightarrow 404 \\ - \ 2 \ 5 \ 5(\text{eight}) \rightarrow - \ 173 \\ \hline 3 \ 4 \ 7(\text{eight}) \rightarrow 231 \end{array}$$

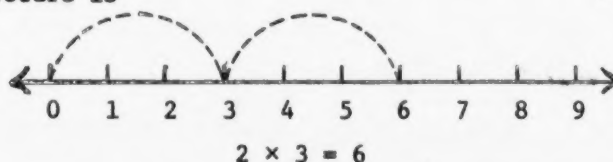
PART I: Help to You in Learning Mathematics

Multiplication as a binary operation: Multiplication is a binary operation on the set of whole numbers --- to each pair of whole numbers a and b , there is assigned an element denoted by $a \times b$ (or $a \cdot b$ or ab). More specifically, to 2 and 3 is assigned 2×3 , or 6, to 3 and 2 is assigned 3×2 , or 6, to 4 and 4 is assigned 4×4 , or 16, and so on. In more traditional terminology, multiplication is an operation on two numbers called *factors* resulting in a unique number called the *product*. Thus, for $4 \times 3 = 12$, 4 and 3 are the factors and 12 is the product. When the factors are known, we *multiply* to find the product.

Using sets to explain multiplication: One way to find the product of two counting numbers is to use addition. Consider 3×2 --- in traditional terminology, 2 is called the *multiplicand* and 3 is called the *multiplier*. To find 3×2 , we can find the sum of three 2's, or $2 + 2 + 2$. This is shown in the following number-line picture.



For 2×3 , 3 is the multiplicand, 2 is the multiplier, and the associated number-line picture is



We may explain multiplication by using disjoint finite equivalent sets. Consider the three sets $\{1, 4\}$, $\{2, 5\}$, and $\{8, 9\}$. They are finite, mutually disjoint (no two of the sets have any members in common), and equivalent (they each have the same number of elements). We have

$$\{1, 4\} \cup \{2, 5\} \cup \{8, 9\} = \{1, 2, 4, 5, 8, 9\}$$

But, since the sets are disjoint,

$$n\{1, 4\} + n\{2, 5\} + n\{8, 9\} = n\{1, 2, 4, 5, 8, 9\}.$$

$$\text{That is, } 2 + 2 + 2 = 6.$$

And, since $3 \times 2 = 2 + 2 + 2$, it follows that $3 \times 2 = 6$.

In other words, the product of two counting numbers a and b is equal to the number of elements in the union of a disjoint sets each having b elements.

The concept of closure: The product of any two counting numbers is a counting number. So, the set of counting numbers is closed with respect to multiplication.

The set of even counting numbers is closed with respect to multiplication (the product of any two even counting numbers is an even counting number).

The set of odd counting numbers is closed with respect to multiplication (the product of any two odd counting numbers is an odd counting number).

Recall that this set is not closed with respect to addition.

The set of whole numbers is closed with respect to multiplication (the product of any two whole numbers is a whole number).

The set of whole numbers that are greater than 10, $\{11, 12, 13, \dots\}$, is closed with respect to multiplication. However, the set of whole numbers that are less than or equal to 10, $\{0, 1, 2, \dots, 10\}$, is *not* closed with respect to multiplication (consider: 2 and 7 are members of the set, but 2×7 , or 14, is not a member of the set).

The Commutative Property of Multiplication: Consider the operation multiplication as defined on the set of whole numbers. $2 \times 3 = 3 \times 2$, $4 \times 7 = 7 \times 4$, $0 \times 9 = 9 \times 0$, and so on. That is, for any whole numbers a and b , the mathematical sentence $a \times b = b \times a$ is true. This illustrates the Commutative Property of Multiplication. In more traditional terminology, the Commutative Property of Multiplication states that the order of the factors may be changed without changing the product.

The Associative Property of Multiplication: Consider the operation multiplication as defined on the set of whole numbers.

$$(2 \times 3) \times 5 = 2 \times (3 \times 5),$$

$$(4 \times 0) \times 2 = 4 \times (0 \times 2),$$

$$(3 \times 3) \times 3 = 3 \times (3 \times 3),$$

and so on.

That is, for any whole numbers, a , b , and c , the mathematical sentence

$$(a \times b) \times c = a \times (b \times c)$$

is true. This illustrates the Associative Property of Multiplication. In more traditional terminology, the Associative Property of Multiplication states that the grouping of the factors may be changed without changing the product.

The multiplicative identity: Consider the operation multiplication as defined on the set of whole numbers. Since 1 is a whole number and since $1 \times a = a \times 1 = a$ for any whole number a , 1 is the *identity element for multiplication* or the *multiplicative identity*.

The Distributive Property: Let $*$ and $\circ$ be two binary operations defined on a set S . If $a * (b \circ c) = (a * b) \circ (a * c)$ for all a, b , and c in S , then $*$ is said to be *left-distributive* with respect to $\circ$. If $(b \circ c) * a = (b * a) \circ (c * a)$ for all a, b , and c in S , then $*$ is said to be *right-distributive* with respect to $\circ$. If $*$ is both left-distributive and right-distributive with respect to $\circ$, then we say, simply, that $*$ is *distributive* with respect to $\circ$. Consider the operations multiplication and addition as defined on the set of whole numbers.

$$3 \times (5 + 2) = (3 \times 5) + (3 \times 2)$$

$$4 \times (0 + 7) = (4 \times 0) + (4 \times 7)$$

$$6 \times (6 + 6) = (6 \times 6) + (6 \times 6)$$

and so on.

That is, for any whole numbers a, b , and c , the mathematical sentence

$$a \times (b + c) = (a \times b) + (a \times c)$$

(often written as $a(b + c) = ab + ac$)

is true. This illustrates that multiplication is left-distributive with respect to addition. Also,

$$(5 + 2) \times 3 = (5 \times 3) + (2 \times 3)$$

$$(0 + 7) \times 4 = (0 \times 4) + (7 \times 4)$$

$$(6 + 6) \times 6 = (6 \times 6) + (6 \times 6)$$

and so on.

That is, for any whole numbers a, b , and c , the mathematical sentence

$$(b + c) \times a = (b \times a) + (c \times a)$$

(often written as $(b + c)a = ba + ca$)

is true. This illustrates that multiplication is right-distributive with respect to addition. Since multiplication is both left-distributive and

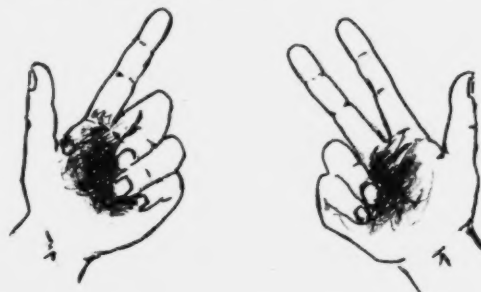
right-distributive with respect to addition, we say, simply that *multiplication is distributive with respect to addition*.

In the next chapter, we will find that although division is right-distributive with respect to addition, it is not left-distributive.

Finger reckoning: "Finger reckoning" or "finger multiplication" is a method for finding multiplication facts involving factors which are each greater than or equal to 5 (7×9 , 8×5 , 6×8 , 7×6 , etc.) As an illustration, consider the following steps for finding 7×8 :

Step 1: "Show" 7 on your left hand by finding the complement of 7 ($10 - 7 = 3$) and closing that many fingers (3), leaving 2 fingers raised.

Step 2: Show 8 on your right hand by closing 2 fingers (the complement of 8 is 2), leaving 3 fingers raised.



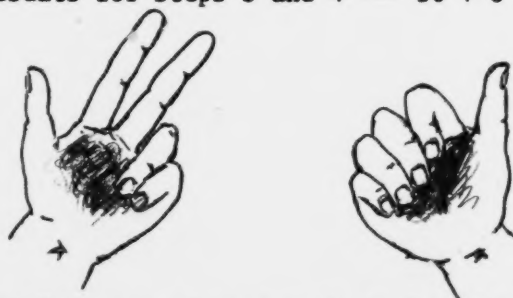
Step 3: Find the sum of the numbers of raised fingers ($2 + 3 = 5$) and multiply that sum by 10 ($10 \times 5 = 50$).

Step 4: Find the product of the numbers of closed fingers ($3 \times 2 = 6$).

Step 5: Find the sum of the results for steps 3 and 4 --- $50 + 6 = 56$. Thus, $7 \times 8 = 56$.

Another illustration:

$$\begin{aligned} 8 \times 6 &= (10 \times (3 + 1)) + (2 \times 4) \\ &= (10 \times 4) + (2 \times 4) \\ &= 40 + 8 \\ &= 48 \end{aligned}$$



Casting out nines: "Casting out nines" is one of many methods for checking multiplication work. (See Chapter 5 if you have forgotten how to find the excess of 9's for a number.)

To check multiplication work, find the excess of 9's for the product of the excesses of 9's for the factors. The result should be the same as the excess of 9's for the product. If it is not, then there is an error in the multiplication work.

Example:

$$\begin{array}{r}
 257 \rightarrow 2 + 5 + 7 = 14 \rightarrow 5 \\
 \times 83 \rightarrow 8 + 3 = 11 \rightarrow 2 \\
 \hline
 771 \\
 2056 \\
 \hline
 21331 \rightarrow 2 + 1 + 3 + 3 + 1 = 10 \rightarrow 1
 \end{array}
 \begin{array}{l}
 \nearrow 5 \times 2 = 10 \rightarrow 1 \\
 \leftarrow 1
 \end{array}$$

equal - so the multiplication work is probably correct

Also,

$$\begin{array}{r}
 257 \div 9 = 28, \text{ remainder } 5 \\
 \times 83 \div 9 = 9, \text{ remainder } 2 \\
 \hline
 771 \\
 2056 \\
 \hline
 21331 \div 9 = 2360, \text{ remainder } 1
 \end{array}
 \begin{array}{l}
 \nearrow 5 \times 2 = 10 \rightarrow 1 \\
 \leftarrow 1
 \end{array}$$

Casting out elevens: "Casting out elevens" is another method for checking multiplication work. (See Chapter 5 if you have forgotten how to find the excess of 11's for a number.)

To check multiplication work, find the excess of 11's for the product of excesses of 11's for the factors. The result should be the same as the excess of 11's for the product. If it is not, then there is an error in the multiplication work.

Example:

$$\begin{array}{r}
 257 \rightarrow (7 + 2) - 5 \rightarrow 9 - 5 = 4 \\
 \times 83 \rightarrow 3 - 8 \rightarrow (3 + 11) - 8 = 6 \\
 \hline
 771 \\
 2056 \\
 \hline
 21331 \rightarrow (1 + 3 + 2) - (3 + 1) \rightarrow 6 - 4 = 2
 \end{array}
 \begin{array}{l}
 \nearrow 4 \times 6 = 24 \rightarrow 2 \\
 \leftarrow 2
 \end{array}$$

Also,

$$\begin{array}{r}
 257 \div 11 = 23, \text{ remainder } 4 \\
 \times 83 \div 11 = 7, \text{ remainder } 6 \\
 \hline
 771 \\
 2056 \\
 \hline
 21331 \div 11 = 1939, \text{ remainder } 2
 \end{array}
 \begin{array}{l}
 \nearrow 4 \times 6 = 24 \rightarrow 2 \\
 \leftarrow 2
 \end{array}$$

Chapter VII

Exercises

- Which does n represent, *a factor* or *the product*?
 a. $47 \times n = 141$
 b. $26 \times 19 = n$
- Copy and complete: The mathematical sentence $7 \times 24 = 24 \times 7$ illustrates the ___?___ Property of Multiplication.
- Copy and complete: The mathematical sentence $(3 \times 7) \times 5 = 3 \times (7 \times 5)$ illustrates the ___?___ Property of Multiplication.
- Suppose a and b represent two nonzero whole numbers and $a \times b = a$. What number does b represent?
- When using "finger reckoning" to find 6×7 , how many fingers should be closed on the hand which shows the factor 6?
- To find 3×12 , we may first rename 12 as $10 + 2$.

	Step
$3 \times 12 = 3 \times (10 + 2)$	(a)
$= (3 \times 10) + (3 \times 2)$	(b)
$= 30 + 6$	(c)
$= 36$	

What property is used in going from step (a) to step (b)?

- Write *Closed* or *Not Closed*.
 a. $\{4, 8, 12, 16, \dots\}$ with respect to multiplication
 b. $\{0, 1\}$ with respect to multiplication
 c. $\{2, 4, 6, 8\}$ with respect to multiplication
- Is addition distributive with respect to multiplication?
- Use "casting out nines" to check the following multiplication work.

$$\begin{array}{r}
 2761 \\
 \times 853 \\
 \hline
 8283 \\
 13705 \\
 \hline
 22088 \\
 \hline
 2354133
 \end{array}$$

10. Use "casting out elevens" to check the following multiplication work.

$$\begin{array}{r} 874 \\ \times 87 \\ \hline 6018 \\ 6992 \\ \hline 75938 \end{array}$$

Chapter VII

Answers for Exercises

1. a. a factor
b. the product
2. The mathematical sentence $7 \times 24 = 24 \times 7$ illustrates the Commutative Property of Multiplication.
3. The mathematical sentence $(3 \times 7) \times 5 = 3 \times (7 \times 5)$ illustrates the Associative Property of Multiplication.
4. 1
5. 4 (the complement of 6 is $10 - 6$, or 4)
6. The Distributive Property (specifically, the left-distributive property of multiplication with respect to addition)
7. a. Closed
b. Closed
c. Not Closed ($2 \times 6 = 12$ and $12 \notin \{2, 4, 6, 8\}$)
8. No (consider: $2 + (3 \times 4) \neq (2 + 3) \times (2 + 4)$)

9.

2761	→	$2 + 7 + 6 + 1 = 16$	→	7	}	$7 \times 7 = 49$	→	4
$\times 853$	→	$8 + 5 + 3 = 16$	→	7			→	4
2354133	→	$2 + 3 + 5 + 4 + 1 + 3 + 3 = 21$	→	3	→	3	→	3

not equal - so there
is an error in the
multiplication work

10. Excess of 11's

847	→	5	}	$5 \times 10 = 50$	→	6
$\times 87$	→	10			→	6
75938	→	5	→	5	→	5

not equal - so there is an
error in the multiplication
work

Chapter VIII

Division

PART I: Help to You in Learning Mathematics

The factors-product relationship: When the factors are known, we *multiply* to find the product. When the product and one factor are known, we *divide* to find the missing factor (or unknown factor).

The four mathematical sentences at the right illustrate the *factors-product* relationship. For each sentence, 32 is the product and 4 and 8 are the factors.

$$32 = 4 \times 8$$

$$32 = 8 \times 4$$

$$32 \div 4 = 8$$

$$32 \div 8 = 4$$

For $105 \div n = 3$, 105 is the product, 3 is the known factor, and n represents the missing factor. To find the missing factor, we divide the product by the known factor. $n = 105 \div 3$.

Look at the vertical form shown at the right for finding $105 \div 3$. In the vertical form for division, the terms dividend, divisor, quotient, and remainder are often used.

Division is sometimes referred to as the operation which "reverses" multiplication, or the operation which is the "inverse" of multiplication, or the operation which "undoes" multiplication.

$$\begin{array}{r} \text{divisor} \\ \downarrow \\ \begin{array}{r} 35 \leftarrow \text{quotient} \\ 3 \overline{)105} \leftarrow \text{dividend} \\ \underline{9} \\ 15 \\ \underline{15} \\ 0 \leftarrow \text{remainder} \end{array} \end{array}$$

Measurement division and partition division: Consider the following problem: John bought 48 strawberry plants. If he places 8 plants in each row, how many rows of strawberry plants will he have? This is an example of "measurement division." In measurement division, we seek "how many groups" when the number in each group is fixed and known. A mathematical sentence for the problem is $n \times 8 = 48$. Thus, $n = 48 \div 8$. Study this work:

$$\begin{array}{r} \text{divisor (number of plants in each row)} \\ \downarrow \\ \begin{array}{r} 6 \leftarrow \text{quotient (number of rows)} \\ 8 \overline{)48} \leftarrow \text{dividend (number of plants in all)} \end{array} \end{array}$$

Note that both the dividend and the divisor are numbers of plants. That is, the dividend and the divisor agree on the kinds of items represented. This is characteristic of measurement division.

Now consider this problem: John bought 48 strawberry plants. If he plants 6 rows of strawberry plants, how many plants will there be in each row? This is an example of "partition division." In partition division, we seek "how many in each group" when the number of groups is fixed and known. A mathematical sentence for the problem is $6 \times n = 48$. Thus, $n = 48 \div 6$.

Study this work:

$$\begin{array}{r} \text{divisor (number of rows)} \\ 8 \div \text{quotient (number of plants in each row)} \\ 6 \overline{)48} \div \text{dividend (number of plants in all)} \end{array}$$

Note that both the dividend and the quotient are numbers of plants. That is, the dividend and the quotient agree on the kinds of items represented. That is characteristic of partition division.

Division as a binary operation: First, let us review the binary operation multiplication.

- (1) The set of whole numbers is closed with respect to multiplication.
- (2) Multiplication is a commutative operation.
- (3) The set of whole numbers contains an identity element for multiplication.
- (4) Multiplication is an associative operation.
- (5) Multiplication is left-distributive with respect to addition. Also, multiplication is right-distributive with respect to addition.

Now let us consider division.

- (1) *The set of whole numbers is not closed with respect to division.*

That is, it is not true that for each pair a and b of whole numbers, $a \div b$ is a whole number. For example, 3 and 5 are whole numbers but $3 \div 5$, or $\frac{3}{5}$, is not a whole number. In later chapters, we will consider sets of nonzero numbers which are closed with respect to division. One such set is the set of nonzero rational numbers.

(2) *Division is not a commutative operation.* That is, it is not true that $a \div b = b \div a$ for all a and b . For example, $3 \div 5 \neq 5 \div 3$.

(3) *Division is not an associative operation.* That is, it is not true that $(a \div b) \div c = a \div (b \div c)$ for all a , b , and c . For example, $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$.

(4) *The set of whole numbers does not contain an identity element for division.* That is, the set of whole numbers does not contain an element e such that $e \div a = a \div e = a$ for each whole number a . The whole number 1 is not such an element because although, in general, $a \div 1 = a$, it is not true that $1 \div a = a \div 1 = a$ for all a . (Since it is true that $a \div 1 = a$, for all a , the number 1 is a *right-identity* element for division.)

(5) *Division is not left-distributive with respect to addition.* That is, it is not true that $a \div (b + c) = (a \div b) + (a \div c)$ for all a , b , and c . For example, $100 \div (20 + 5) \neq (100 \div 20) + (100 \div 5)$. However, it is true that $(b + c) \div a = (b \div a) + (c \div a)$ provided $a \neq 0$. Thus, for nonzero numbers, division is right-distributive with respect to addition.

The principle of compensation: When doing division work, we may use the principle of compensation --- if both the product and the known factor are multiplied by the same nonzero number, then the unknown factor remains unchanged. This principle will be useful in a later chapter when we divide with decimals.

Casting out nines: "Casting out nines" is one of many methods for checking division work. (See Chapter 5 if you have forgotten how to find the excess of 9's for a number.)

To check "exact division" work (division work where the remainder is 0), find the excess of 9's for the product of the excesses of 9's for the divisor and the quotient. The result should be the same as the excess of 9's for the dividend. If it is not, then there is an error in the division work.

Example:

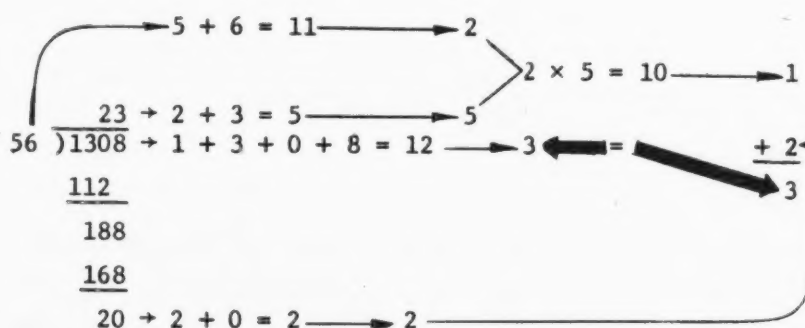
$7 + 8 = 15 \rightarrow 6$
 $46 \div 4 + 6 = 10 \rightarrow 1$
 $78)3588 \rightarrow 3 + 5 + 8 + 8 = 24 \rightarrow 6$
 $6 \times 1 = 6 \rightarrow 6$
 $6 = 6$

$$\begin{array}{r}
 312 \\
 78 \overline{)3588} \\
 \underline{468} \\
 468 \\
 \underline{468} \\
 0
 \end{array}$$

equal - so the division work is probably correct

To check "inexact division" work (division work where the remainder is not 0), find the excess of 9's for the product of the excess of 9's for the divisor and quotient, add that excess to the excess of 9's for the remainder, and then find the excess of 9's for that sum. The result should be the same as the excess of 9's for the dividend. If it is not, then there is an error in the division work.

Example:



Divisibility tests: A whole number is *divisible by 2* if and only if the digit in one's place is 0, 2, 4, 6, or 8. Thus, each of the five numbers 26, 38, 72, 80, and 1,554 is divisible by 2 but 567 is not divisible by 2.

A whole number is *divisible by 3* if and only if the excess of 9's for the number is divisible by 3. Thus, 5,478 is divisible by 3 because 6, the excess of 9's for 5,478 ($5 + 4 + 7 + 8 = 24 \rightarrow 6$), is divisible by 3. The number 371 is not divisible by 3 because 2, the excess of 9's for 371, is not divisible by 3.

A whole number is *divisible by 4* if and only if the number "represented" by the digits in ten's place and one's place is divisible by 4. For example, 5,732 is divisible by 4 because the number 32 is divisible by 4. The number 63,627 is not divisible by 4 because 27 is not divisible by 4.

A whole number is *divisible by 5* if and only if the digit in one's place is 0 or 5. Thus, each of 270 and 1,285 is divisible by 5 but 234 is not divisible by 5.

A whole number is *divisible by 9* if and only if the excess of 9's for the number is 0. Thus, 2,493 is divisible by 9 because the excess of 9's for 2,493 is 0 ($2 + 4 + 9 + 3 = 18 \rightarrow 0$). The number 6,281 is not divisible by 9 because the excess of 9's for 6,281 is not 0 ($6 + 2 + 8 + 1 = 17 \rightarrow 8 \neq 0$).

Division work with base-five numerals:

We may use the table at the right to find basic division facts for base five:

$$4 \div 2 = 2,$$

$$13 \div 4 = 2,$$

$$22 \div 3 = 4,$$

and so on (remember - we are working in base five).

Now let us find $41_{\text{five}} \div 3_{\text{five}}$. Think of 41_{five} as 4 fives and 1 one.

Divide fives: The greatest multiple of 3 that can be subtracted from 4 is 3. $3 = 1 \times 3$ so we write a 1 in five's place in the numeral for the quotient. There is 1 five left to be divided. Rename 1 five as 10 ones.

Divide ones: There are 11 ones to be divided. The greatest multiple of 3 that can be subtracted from 11 is 11. $11 = 2 \times 3$ so we write a 2 in one's place in the numeral for the quotient. The remainder is 0.

$$\text{Thus, } 41_{\text{five}} \div 3_{\text{five}} = 12_{\text{five}}.$$

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	11	13
3	3	11	14	22
4	4	13	22	31

Base Five

$$\begin{array}{r} 12 \\ 3 \overline{) 41} \\ \underline{3} \\ 11 \\ \underline{11} \\ 0 \end{array}$$

Base Five

Chapter VIII

Exercises

1. Which does n represent, *the unknown factor* or *the product*? $312 \div n = 3$
2. When the product and one factor are known, what operation is used to find the unknown factor?
3. Copy and complete: The mathematical sentence $5 \div 2 \neq 2 \div 5$ illustrates that division is not ? .
4. Copy and complete: The mathematical sentence $(40 \div 4) \div 2 \neq 40 \div (4 \div 2)$ illustrates that division is not ? .
5. Which does the following problem involve, *measurement division* or *partition division*? A book store packed 300 books in 15 cartons with the same number of books in each carton. How many books were there in each carton?
6. Is 412,301,025 divisible by 3? (Use the test for divisibility by 3.)
7. Thinking of the test for divisibility by 4, copy and complete the following: The number 87,716 is divisible by 4 because ? is divisible by ? .
8. Use "casting out nines" to check the following division work.

$$\begin{array}{r}
 111 \\
 84 \overline{)9382} \\
 \underline{84} \\
 98 \\
 \underline{84} \\
 152 \\
 \underline{84} \\
 68
 \end{array}$$

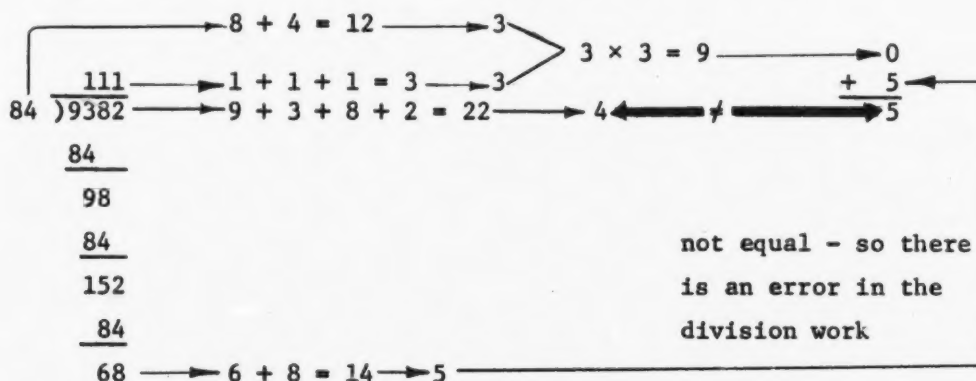
9. Copy and complete
 - a. $14_{\text{five}} \div 3_{\text{five}} = \text{ ? }$
 - b. $14_{\text{eight}} \div 3_{\text{eight}} = \text{ ? }$
10. If $a \div b = c$ and $k \neq 0$, then $(k \times a) \div (k \times b) = c$. This illustrates what principle?

Chapter VIII

Answers for exercises

1. the unknown factor
2. Division
3. The mathematical sentence $5 + 2 \neq 2 + 5$ illustrates that division is not commutative.
4. The mathematical sentence $(40 + 4) + 2 \neq 40 + (4 + 2)$ illustrates that division is not associative.
5. partition division
6. Yes (the excess of 9's is 18, which is divisible by 3)
7. The number 87,716 is divisible by 4 because 16 is divisible by 4.

8.



9. a. $14_{\text{five}} + 3_{\text{five}} = 3_{\text{five}}$
 b. $14_{\text{eight}} + 3_{\text{eight}} = 4_{\text{eight}}$
10. principle of compensation

PART I: Help to You in Learning Mathematics

Inverse elements: Let $*$ be a binary operation defined on a set S . Furthermore, let S contain an identity element e for the operation $*$. Now if a and b are elements of S such that $a * b = b * a = e$ then a and b are mutually inverse. That is, a is the inverse of b with respect to $*$ and b is the inverse of a with respect to $*$. The result of the operation on two inverse elements has to be the identity element for that operation. The *additive inverse* of 2 is the integer -2 because $(-2) + 2 = 2 + (-2) = 0$ and 0 is the additive identity. The *multiplicative inverse* of 2 is $\frac{1}{2}$ because $\frac{1}{2} \times 2 = 2 \times \frac{1}{2} = 1$ and 1 is the multiplicative identity.

Some elements are self-inverse with respect to an operation. For example, 0 is self-inverse with respect to addition, 1 is self-inverse with respect to multiplication, and -1 is self-inverse with respect to multiplication.

The set of integers: The set of natural numbers (counting numbers) does not contain an additive identity. The set of whole numbers, $\{0, 1, 2, 3, \dots\}$, contains an additive identity but does not contain an additive inverse for each of its elements. Let us now consider a set which does contain an additive inverse for each of its elements -- the set of *integers*. The set of integers may be thought of as the union of $\{1, 2, 3, \dots\}$, $\{0\}$, and $\{-1, -2, -3, \dots\}$ where -1 is the additive inverse of 1, -2 is the additive inverse of 2, and so on. (Recall that 0 is self-inverse with respect to addition.) Thus, the set of integers is

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Sometimes the set of integers is described as $\{0, \pm 1, \pm 2, \pm 3, \dots\}$ where $\pm a$ is simply a way of indicating that both a and $-a$ are in the set. Some capital letters that are used to denote the set of integers are $\mathbb{Z}$ (quite popular), $\mathbb{I}$, and $\mathbb{J}$. Also, integers are sometimes referred to as "signed numbers" -- in which case 1 is expressed as $+1$, 2 as $+2$, and so on. Sometimes the signs "+" and "-" are raised. For example, $^+1$ and $^-1$.

The integers 1, 2, 3, $\dots$ are the *positive integers*. The integers -1 , -2 , -3 , $\dots$ are the *negative integers*. The integer 0 is neither positive

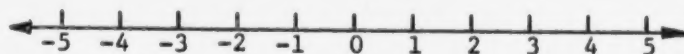
nor negative. Thus, the set of *non-negative integers* is $\{0, 1, 2, 3, \dots\}$

and the set of *non-positive integers* is $\{0, -1, -2, -3, \dots\}$.

The set of *even integers* is $\{\dots, -4, -2, 0, 2, 4, \dots\}$

while the set of *odd integers* is $\{\dots, -4, -2, 0, 2, 4, \dots\}$.

Also, we may use a number-line picture to show the set of integers.



In a number-line picture, if the point shown for one number is to the left of the point shown for a second number, then the first number is less than the second number. Thus, -3 is less than -2 , -4 is less than -1 , -5 is less than 3 , and so on.

Addition of integers: Addition is a binary operation defined on the set of integers -- to each pair of integers a and b , there is assigned an element denoted by $a + b$. More specifically, to 5 and -3 is assigned $5 + (-3)$, or 2 , to -3 and 5 is assigned $(-3) + 5$, or 2 , to 4 and -8 is assigned $4 + (-8)$, or -4 , to -2 and -3 is assigned $(-2) + (-3)$, or -5 , and so on.

$$5 + (-3) = 2 \qquad +5 + -3 = +2$$

$$(-3) + 5 = 2 \qquad -3 + +5 = +2$$

$$4 + (-8) = -4 \qquad +4 + -8 = -4$$

$$(-2) + (-3) = -5 \qquad -2 + -3 = -5$$

The sum of any two integers is always an integer -- the set of integers is closed with respect to addition.

Addition is a commutative operation. That is, for any integers a and b , the mathematical sentence $a + b = b + a$ is true.

Addition is an associative operation. That is, for any integers a , b , and c , the mathematical sentence $(a + b) + c = a + (b + c)$ is true.

The integer 0 is the additive identity for the set of integers. That is, $0 + a = a + 0 = a$ for any integer a .

Also, each integer has an additive inverse which is in the set of integers.

Subtraction of integers: The set of integers is closed with respect to subtraction. That is, for any integers a and b , $a - b$ is an integer.

$5 - 3 = 2$	$+5 - +3 = +2$
$3 - 5 = -2$	$+3 - +5 = -2$
$(-2) - 7 = -9$	$-2 - +7 = -9$
$3 - (-4) = 7$	$+3 - -4 = +7$
$(-8) - (-9) = 1$	$-8 - -9 = +1$

Subtraction is not a commutative operation. Consider: $5 - 3 \neq 3 - 5$

Subtraction is not an associative operation. Consider: $(2 - 3) - 5 \neq 2 - (3 - 5)$

The set of integers does not contain an identity element for subtraction. However, the integer 0 is a right-identity for subtraction -- that is, $a - 0 = a$ for any integer a .

Multiplication of integers: Multiplication is a binary operation defined on the set of integers -- to each pair of integers a and b , there is assigned an element denoted by $a \times b$ (or $a \cdot b$ or ab). More specifically, to 3 and 7 is assigned 3×7 , or 21, to -4 and 2 is assigned $(-4) \times 2$, or -8 , to 6 and -5 is assigned $6 \times (-5)$, or -30 , to -3 and -9 is assigned $(-3) \times (-9)$, or 27, and so on.

$3 \times 7 = 21$	$+3 \times +7 = +21$
$(-4) \times 2 = -8$	$-4 \times +2 = -8$
$6 \times (-5) = -30$	$+6 \times -5 = -30$
$(-3) \times (-9) = 27$	$-3 \times -9 = +27$

The product of any two integers is always an integer -- the set of integers is closed with respect to multiplication.

Multiplication is a commutative operation. That is, for any integers a and b , the mathematical sentence $a \times b = b \times a$ is true.

Multiplication is an associative operation. That is, for any integers a , b , and c , the mathematical sentence $(a \times b) \times c = a \times (b \times c)$ is true.

The integer 1 is the multiplicative identity for the set of integers. That is, $1 \times a = a \times 1 = a$ for any integer a .

The set of integers contains the multiplicative inverses of 1 and -1 (the multiplicative inverse of 1 is 1; of -1 is -1), but does not contain multiplicative inverses for the other integers.

Multiplication is distributive with respect to addition. That is, for any integers a , b , and c , the mathematical sentences

$$a \times (b + c) = (a \times b) + (a \times c)$$

and

$$(b + c) \times a = (b \times a) + (c \times a)$$

are true.

Division of integers: The set of integers is not closed with respect to division. That is, it is not true that for each pair a and b of integers, $a \div b$ is an integer. For example, 3 and 5 are integers but $3 \div 5$, or $\frac{3}{5}$, is not an integer.

Division is not a commutative operation. Consider: $3 \div 5 \neq 5 \div 3$

Division is not an associative operation. Consider: $(2 \div 3) \div 4 \neq 2 \div (3 \div 4)$

The set of integers does not contain an identity element for division. However, the integer 1 is a right-identity for division -- that is, $a \div 1 = a$ for any integer a .

Division is not left-distributive with respect to addition. However, $(b + c) \div a = (b \div a) + (c \div a)$ provided $a \neq 0$.

Chapter X

Exercises

1. Write *Closed* or *Not Closed*.
 - a. The set of even integers with respect to addition
 - b. The set of even integers with respect to multiplication
 - c. The set of odd integers with respect to addition
 - d. The set of odd integers with respect to multiplication
2. The integer -8 is less than -3, so in a horizontal number-line picture, the point for -3 would be shown to which, *the left* or *the right*, of the point for -8?
3. Name the two integers which are self-inverse with respect to multiplication.
4. Is it true that the union of the set of even integers and the set of odd integers is the set of integers?
5. Is it true that the union of the set of positive integers and the set of negative integers is the set of integers?
6. For the set of integers,
 - a. the additive identity is ? .
 - b. the multiplicative identity is ? .
7. Copy and complete:
 - a. The additive inverse of 32 is ? .
 - b. The additive inverse of -14 is ? .
8. Copy and complete: The mathematical sentence
$$3 \div (4 + 5) \neq (3 \div 4) + (3 \div 5)$$
illustrates that division is not ? -distributive with respect to addition.
9. The sum of two negative integers is which, a *positive integer* or a *negative integer*?
10. The product of two negative integers is which, a *positive integer* or a *negative integer*?

Chapter X

Answers for exercises

1. a. Closed
b. Closed
c. Not Closed
d. Closed
2. The right
3. -1, 1
4. Yes
5. No (0 would not be in the set)
6. a. 0
b. 1
7. a. The additive inverse of 32 is -32.
b. The additive inverse of -14 is 14.
8. The mathematical sentence $3 + (4 + 5) \neq (3 + 4) + (3 + 5)$ illustrates that division is not left distributive with respect to addition.
9. a negative integer
10. a positive integer

Chapter XI

Fractions

PART I: Help to You in Learning Mathematics

Fractions: The picture in the box at the right is called a *fraction*. The symbol above the bar names the *numerator* of the fraction. The symbol below the bar names the *denominator* of the fraction. A fraction such as $\frac{2}{3}$ may be used in many ways. Four ways are:

$$\frac{2}{3}$$

- (1) To express "what part of"
- (2) As a name for a rational number
- (3) To indicate division
- (4) To express a ratio

Using a fraction to express "what part of": The fraction $\frac{2}{3}$ may be used to express *what part of* the diagram is shaded for each of a - c below.



a



b



c

Diagrams a and b suggest the traditional "parts of a whole" idea while diagram c suggests the traditional "parts of a group" idea.

Using a fraction as a name for a rational number: Any fraction of the form $\frac{p}{q}$ where p and q name integers ($q \neq 0$) can be considered a name for a *rational number*. Conversely, every rational number can be named with a fraction of this form.

So, $\frac{2}{3}$ names the rational number which is the solution for $3x = 2$. We shall study the system of rational numbers in detail later in this chapter.

Using a fraction to indicate division: The fraction $\frac{2}{3}$ may be used to indicate "2 divided by 3" or " $2 \div 3$." Consider the rational number that is named by the fraction $\frac{2}{3}$. In the next chapter, we will find that one way to rename the number with a decimal is to think of $\frac{2}{3}$ as indicating "2 divided by 3" and then do the division work.

Using a fraction to express a ratio: At the right, 5 triangular regions are shown, 2 of which are shaded. The ratio of the number of shaded regions to the number of unshaded regions is 2 to 3 or 2:3. This ratio may also be expressed with the fraction $\frac{2}{3}$. We shall study the concept of ratio in detail in a later chapter.



The set of rational numbers: The set of integers contains an additive inverse for each of its elements. However, the set of integers does not contain a multiplicative inverse for each of its nonzero elements. For example, the set of integers does not contain a multiplicative inverse for the integer 2. Let us now consider a set which does contain a multiplicative inverse for each of its nonzero elements -- the set of *rational numbers*. The set of rational numbers may be thought of as all numbers which can be expressed in the form $\frac{p}{q}$, where p and q name integers ($q \neq 0$). Thus, $\frac{2}{3}$, $\frac{9}{5}$, and $\frac{-4}{8}$ are rational numbers. Also, the numbers 0, 15, and -2 are rational numbers because they can be expressed as $\frac{0}{1}$, $\frac{15}{1}$, and $\frac{-2}{1}$ (that is, in the form $\frac{p}{q}$, where p and q name integers ($q \neq 0$)). There are many numbers which you have probably worked with, but which are not rational numbers. For example, $\sqrt{2}$ and π are not rational numbers -- they cannot be expressed in the form $\frac{p}{q}$, where p and q name integers, $q \neq 0$. (The reader is reminded that $\frac{22}{7}$ does not name the number π -- it names a number which is *approximately equal to* π .) Numbers such as $\sqrt{2}$ and π are called *irrational numbers*. The union of the set of rational numbers and the set of irrational numbers is the set of *real numbers*.

Rational numbers have many names: Numerals which name the same number are said to be equivalent. If $\frac{p}{q}$ and $\frac{r}{s}$ represent rational numbers, then

$$\frac{p}{q} = \frac{r}{s} \text{ if and only if } ps = qr.$$

$\frac{p}{q} = \frac{r}{s}$ means that the numbers $\frac{p}{q}$ and $\frac{r}{s}$ are equal and that the fractions $\frac{p}{q}$ and $\frac{r}{s}$ are equivalent. For example, $\frac{3}{5} = \frac{6}{10}$ because $3 \times 10 = 5 \times 6$. The numbers $\frac{3}{5}$ and $\frac{6}{10}$ are equal. The fractions $\frac{3}{5}$ and $\frac{6}{10}$ are equivalent.

There are many kinds of numerals for rational numbers: standard numerals, exponent forms, fractions, decimals, mixed forms, and so on. The rational

number *thirty-two* has many names: the standard numeral 32, the exponent form 2^5 , and so on. The rational number *two and one tenth* has many names: the fraction $\frac{21}{10}$, the decimal 2.1, the mixed form $2\frac{1}{10}$, and so on. In this chapter we shall do arithmetic work with fractions. In the next chapter, we shall do arithmetic work with decimals.

Simplest form: A fraction is in *simplest form* if the fraction shows a numerator and a denominator which do not have any number greater than 1 as a common factor. Thus, the fractions $\frac{1}{2}$, $\frac{2}{5}$, and $\frac{3}{7}$ are in simplest form while the fractions $\frac{2}{4}$, $\frac{6}{15}$, and $\frac{21}{49}$ are not in simplest form. Two numbers are *relatively prime* if the greatest common factor of the two numbers is 1. So, we can say that a fraction is in simplest form if the numerator and the denominator are relatively prime.

Addition of rational numbers: Addition is a binary operation defined on the set of rational numbers -- to each pair of rational numbers a and b , there is assigned an element denoted by $a + b$. More specifically, to 3 and 8 is assigned $3 + 8$, or 11, to $\frac{1}{4}$ and $\frac{1}{3}$ is assigned $\frac{1}{4} + \frac{1}{3}$, or $\frac{7}{12}$, to $\frac{1}{3}$ and $\frac{1}{4}$ is assigned $\frac{1}{3} + \frac{1}{4}$, or $\frac{7}{12}$, and so on.

The sum of any two rational numbers is always a rational number -- the set of rational numbers is closed with respect to addition.

Addition is a commutative operation. That is, for any rational numbers a and b , the mathematical sentence $a + b = b + a$ is true.

Addition is an associative operation. That is, for any rational numbers a , b , and c , the mathematical sentence $(a + b) + c = a + (b + c)$ is true.

The rational number 0 is the additive identity for the set of rational numbers. That is, $0 + a = a + 0 = a$ for any rational number a .

Also, each rational number has a additive inverse which is in the set of rational numbers. The additive inverse of 0 is 0, of 3 is -3, of $\frac{2}{3}$ is $-\frac{2}{3}$, of $-\frac{7}{5}$ is $\frac{7}{5}$, and so on.

Subtraction of rational numbers: The set of rational numbers is closed with respect to subtraction. That is, for any rational numbers a and b , $a - b$ is a rational number.

Subtraction is neither a commutative operation nor an associative operation.

The set of rational numbers does not contain an identity element for subtraction. However, the rational number 0 is a right-identity for subtraction -- that is, $a - 0 = a$ for any rational number a .

Multiplication of rational numbers: Multiplication is a binary operation defined on the set of rational numbers -- to each pair of rational numbers a and b , there is assigned an element denoted by $a \times b$ (or $a \cdot b$ or ab). More specifically, to 3 and 8 is assigned 3×8 , or 24, to $\frac{1}{4}$ and $\frac{1}{3}$ is assigned $\frac{1}{4} \times \frac{1}{3}$, or $\frac{1}{12}$, to $\frac{1}{3}$ and $\frac{1}{4}$ is assigned $\frac{1}{3} \times \frac{1}{4}$, or $\frac{1}{12}$, and so on.

The product of any two rational numbers is always a rational number -- the set of rational numbers is closed with respect to multiplication.

Multiplication is a commutative operation. That is, for any rational numbers a and b , the mathematical sentence $a \times b = b \times a$ is true.

Multiplication is an associative operation. That is, for any rational numbers a , b , and c , the mathematical sentence $(a \times b) \times c = a \times (b \times c)$ is true.

The rational number 1 is the multiplicative identity for the set of rational numbers. That is, $1 \times a = a \times 1 = a$ for any rational number a .

Also, each nonzero rational number has a multiplicative inverse which is in the set of rational numbers. The multiplicative inverse of 1 is 1, of -1 is -1, of 2 is $\frac{1}{2}$, of $\frac{5}{9}$ is $\frac{9}{5}$, and so on. Recall that two numbers are mutually inverse with respect to an operation if the result of the operation on the two elements is the identity element for the operation. The number 0 does not have a multiplicative inverse -- that is, there is no number such that the product of that number and 0 is equal to 1.

Multiplication is distributive with respect to addition. That is, for any rational numbers a , b , and c , the following sentences are true.

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$(b + c) \times a = (b \times a) + (c \times a)$$

Division of rational numbers: The set of rational numbers *is not* closed with respect to division. That is, it is not true that for each pair a and b of rational numbers, $a \div b$ is a rational number. For example, 3 and 0 are rational numbers but $3 \div 0$ is not a rational number. In fact, $3 \div 0$ is not defined. However, the set of nonzero rational numbers is closed with respect to division. Also, if a represents any rational number and b represents any nonzero rational number, then $a \div b$ is a rational number.

Division is neither a commutative operation nor an associative operation.

The set of rational numbers does not contain an identity element for division. However, the rational number 1 is a right-identity for division -- that is, $a \div 1 = a$ for any rational number a .

Division is not left-distributive with respect to addition. However, $(b + c) \div a = (b \div a) + (c \div a)$ provided $a \neq 0$.

Factors of a number: If a whole number a can be expressed as a product of two whole numbers b and c -- $a = b \times c$, then b and c are *factors* of the number a . For example, 4 is a factor of 20 because $20 = 4 \times 5$ (which means that 5 is also a factor). The set of factors of 20 is $\{1, 2, 4, 5, 10, 20\}$, of 5 is $\{1, 5\}$, of 30 is $\{1, 2, 3, 5, 6, 10, 15, 30\}$, of 13 is $\{1, 13\}$, and so on. The number 1 is a factor of any whole number. Also, any whole number is a factor of itself. Thus, every whole number greater than 1 has at least two factors, 1 and the whole number itself.

Expressing numbers as products of primes: A whole number greater than 1 that has only two factors, 1 and itself, is called a *prime number* or simply, a *prime*. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. A whole number greater than 1 that is not a prime number is called a *composite number*.

Any composite number can be expressed as a product of primes in only one way if the order in which the factors appear is not considered. "Factor trees" are useful in expressing a composite as a product of primes:



Thus,

$$90 = 2 \times 3 \times 3 \times 5$$

and

$$200 = 2 \times 2 \times 2 \times 5 \times 5.$$

Using exponent forms, we have

$$90 = 2 \times 3^2 \times 5$$

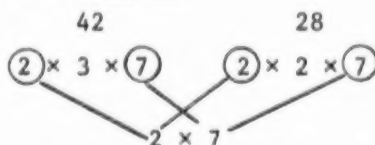
$$200 = 2^3 \times 5^2.$$

G. C. F. and L. C. M.: The *greatest common factor* (G. C. F.) of two or more whole numbers is the greatest whole number which is a factor of each of them. For example, the G. C. F. of 42 and 28 is 14. We may express this by writing $GCF(42, 28) = 14$. The G. C. F. of 42, 28, and 21 is 7 -- that is, $GCF(42, 28, 21) = 7$.

We may use sets to find the G. C. F. of two numbers. For example, to find $\text{GCF}(42, 28)$, let $A = \{1, 2, 3, 6, 7, 14, 21, 42\}$, the set of factors of 42, and let $B = \{1, 2, 4, 7, 14, 28\}$, the set of factors of 28. Then, $\text{GCF}(42, 28)$ is the greatest member of $\{1, 2, 7, 14\}$, the *intersection* of sets A and B. (The intersection of sets A and B is the set of common factors of 42 and 28.) Thus, $\text{GCF}(42, 28) = 14$.

A second way to find the G. C. F. of two numbers: Express each number as a product of primes. Then write a product expression for the G. C. F. by using as factors all common prime factors -- use each as a factor the *least* number of times that it appears in the product-of-primes expressions.

Example 1: $\text{GCF}(42, 28) = \underline{\quad? \quad}$



So, $\text{GCF}(42, 28) = 2 \times 7$, or 14.

Example 2: $\text{GCF}(84, 60) = \underline{\quad? \quad}$



So, $\text{GCF}(84, 60) = 2 \times 2 \times 3$, or 12.

A third way to find the G. C. F. of two numbers is to use the "Euclidean algorithm." First, the greater number is divided by the lesser number. Then in each succeeding step, the previous divisor is divided by the previous remainder. This continues until a remainder of 0 is obtained. The last nonzero remainder is the G. C. F. of the two numbers.

Example 1: $\text{GCF}(42, 28) = \underline{\quad? \quad}$

$$42 \div 28 = 1, \text{ remainder } (14)$$

$$28 \div 14 = 2, \text{ remainder } 0$$

$$\text{GCF}(42, 28) = 14$$

Example 2: $\text{GCF}(84, 60) = \underline{\quad? \quad}$

$$84 \div 60 = 1, \text{ remainder } 24$$

$$60 \div 24 = 2, \text{ remainder } (12)$$

$$24 \div 12 = 2, \text{ remainder } 0$$

$$\text{GCF}(84, 60) = 12$$

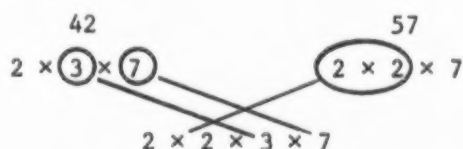
NOTE: Since the Euclidean algorithm involves division, the G. C. F. is often referred to as the G. C. D. (greatest common divisor). For example, $\text{GCD}(42, 28) = 14$.

The *least common multiple* (L. C. M.) of two or more whole numbers is the least counting-number multiple of the numbers. For example, the L. C. M. of 42 and 28 is 84. We may express this by writing $\text{LCM}(42, 28) = 84$. The L. C. M. of 15, 12, and 8 is 120 -- that is, $\text{LCM}(15, 12, 8) = 120$.

We may use sets to find the L. C. M. of two numbers. For example, to find $\text{LCM}(42, 28)$, let $A = \{42, 84, 126, 168, \dots\}$, the set of counting-number multiples of 42, and let $B = \{28, 56, 84, 112, 140, 168, \dots\}$, the set of counting-number multiples of 28. Then $\text{LCM}(42, 28)$ is the least member of $\{84, 168, \dots\}$, the *intersection* of sets A and B. (The intersection of sets A and B is the set of common counting-number multiples of 42 and 28.) Thus, $\text{LCM}(42, 28) = 84$.

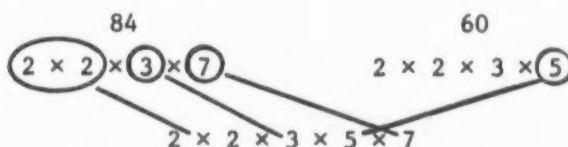
A second way to find the L. C. M. of two numbers: Express each number as a product of primes. Then write a product expression for the L. C. M. by using as factors all primes that appear -- use each as a factor the *greatest* number of times that it appears in the product-of-primes expressions.

Example 1: $\text{LCM}(42, 28) = \underline{\quad? \quad}$



So, $\text{LCM}(42, 28) = 2 \times 2 \times 3 \times 7$, or 84.

Example 2: $\text{LCM}(84, 60) = \underline{\quad? \quad}$



So, $\text{LCM}(84, 60) = 2 \times 2 \times 3 \times 5 \times 7$, or 420.

A third way to find the L. C. M. of two numbers is to divide the product of the two numbers by the G. C. F. of the two numbers.

For example, $\text{GCF}(42, 28) = 14$, so

$$\begin{aligned}\text{LCM}(42, 28) &= (42 \times 28) \div 14 \\ &= 1176 \div 14 \\ &= 84.\end{aligned}$$

Since the G. C. F. is a factor of either number, it is easier to divide one of the two numbers by the G. C. F. and then multiply the other number by the resulting quotient.

For example, $\text{GCF}(42, 28) = 14$, so

$$\begin{aligned}\text{LCM}(42, 28) &= (42 \div 14) \times 28 \\ &= 3 \times 28 \\ &= 84.\end{aligned}$$

Also, $\text{LCM}(42, 28) = 42 \times (28 \div 14)$

$$\begin{aligned}&= 42 \times 2 \\ &= 84.\end{aligned}$$

Adding with fractions: Let $\frac{p}{q}$ and $\frac{r}{s}$ represent rational numbers. Then

$$\frac{p}{q} + \frac{r}{s} = \frac{(p \times s) + (q \times r)}{q \times s}.$$

For example,

$$\begin{aligned}\frac{1}{6} + \frac{2}{3} &= \frac{(1 \times 3) + (6 \times 2)}{6 \times 3} \\ &= \frac{3 + 12}{18} \\ &= \frac{15}{18}.\end{aligned}$$

Then $\frac{15}{18}$ can be renamed with a fraction in simplest form by dividing both 15, the numerator, and 18, the denominator, by the G. C. F. of 15 and 18.

$$\text{So, } \frac{1}{6} + \frac{2}{3} = \frac{5}{6}.$$

If, when adding with fractions, the fractions show the same denominator, then the "common-denominator method" may be used. If $\frac{a}{c}$ and $\frac{b}{c}$ represent rational numbers, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

For example,

$$\begin{aligned}\frac{3}{5} + \frac{1}{5} &= \frac{3+1}{5} \\ &= \frac{4}{5}\end{aligned}$$

If the fractions do not show the same denominator, then the addends may be renamed so that the common-denominator method may be used. Consider $\frac{1}{6} + \frac{2}{3}$. In renaming, 18, the product of the denominators could be used as the common denominator, but the *least common denominator* is preferred. In general the least common denominator of $\frac{p}{q}$ and $\frac{r}{s}$ is the least common multiple of q and s . So, 6 is the least common denominator of $\frac{1}{6}$ and $\frac{2}{3}$. Thus, to find $\frac{1}{6} + \frac{2}{3}$, we may rename --

$$\begin{aligned}\frac{1}{6} + \frac{2}{6} &= \frac{1}{6} + \frac{4}{6} \\ &= \frac{5}{6}\end{aligned}$$

Subtracting with fractions: Let $\frac{p}{q}$ and $\frac{r}{s}$ represent rational numbers. Then

$$\frac{p}{q} - \frac{r}{s} = \frac{(p \times s) - (q \times r)}{q \times s}$$

For example,

$$\begin{aligned}\frac{5}{7} - \frac{2}{3} &= \frac{(5 \times 3) - (7 \times 2)}{7 \times 3} \\ &= \frac{15 - 14}{7 \times 3} \\ &= \frac{1}{21}\end{aligned}$$

Also, if $\frac{a}{c}$ and $\frac{b}{c}$ represent rational numbers, then

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}.$$

Thus,

$$\begin{aligned}\frac{5}{7} - \frac{2}{3} &= \frac{15}{21} - \frac{14}{21} \\ &= \frac{1}{21}.\end{aligned}$$

Multiplying with fractions: Let $\frac{p}{q}$ and $\frac{r}{s}$ represent rational numbers. Then

$$\frac{p}{q} \times \frac{r}{s} = \frac{p \times r}{q \times s}.$$

For example,

$$\begin{aligned}\frac{2}{3} \times \frac{5}{7} &= \frac{2 \times 5}{3 \times 7} \\ &= \frac{10}{21}.\end{aligned}$$

Often, multiplication work may be simplified by using what is traditionally called "cancellation."

For example,

$$\frac{\cancel{3}}{4} \times \frac{5}{\cancel{21}_7} = \frac{5}{28}.$$

Also,

$$\frac{\cancel{2}}{\cancel{3}_1} \times \frac{\cancel{15}_3}{\cancel{18}_6} = \frac{3}{8}.$$

Dividing with fractions: If, when dividing with fractions, the fractions show the same denominator, then the "common-denominator method" may be used.

If $\frac{a}{c}$ and $\frac{b}{c}$ represent rational numbers ($\frac{b}{c} \neq 0$), then

$$\frac{a}{c} \div \frac{b}{c} = a \div b.$$

For example,

$$\begin{aligned}\frac{4}{7} \div \frac{6}{7} &= 4 \div 6 \\ &= \frac{4}{6}, \text{ or } \frac{2}{3}.\end{aligned}$$

If the fractions do not show the same denominator, then the product and the known factor may be renamed so that the common-denominator method may be used.

For example,

$$\begin{aligned}\frac{2}{3} \div \frac{4}{5} &= \frac{10}{15} \div \frac{12}{15} \\ &= 10 \div 12 \\ &= \frac{10}{12}, \text{ or } \frac{5}{6}.\end{aligned}$$

A more popular method, however, is the "reciprocal method." If $\frac{p}{q}$ and $\frac{r}{s}$ represent rational numbers ($\frac{r}{s} \neq 0$), then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \times \frac{s}{r}.$$

That is, to find the unknown factor, the product is multiplied by the reciprocal (multiplicative inverse) of the known factor.

For example,

$$\begin{aligned}\frac{2}{3} \div \frac{5}{7} &= \frac{2}{3} \times \frac{7}{5} \\ &= \frac{14}{15}.\end{aligned}$$

1. Write a fraction to indicate what part of this diagram is shaded.



2. Are 14 and 25 relatively prime?
3. Express the number as a product of primes.
 - a. 142
 - b. 825
 - c. 256
4. Using the Euclidean algorithm, find the greatest common factor of 216 and 81.
5. Find the least common multiple of 216 and 81.
6. Use the common-denominator method to find
 - a. $\frac{2}{15} + \frac{5}{15}$.
 - b. $\frac{1}{4} + \frac{4}{9}$.
7. Use the common-denominator method to find
 - a. $\frac{7}{8} - \frac{5}{8}$.
 - b. $\frac{23}{25} - \frac{4}{5}$.
8. Find the product of $\frac{7}{10}$ and $\frac{20}{21}$. Express your answer with a fraction in simplest form.
9. Use the common-denominator method to find
 - a. $\frac{12}{15} + \frac{4}{15}$.
 - b. $\frac{3}{7} + \frac{33}{35}$.
10. Use the reciprocal method to find
 - a. $\frac{3}{4} + \frac{9}{16}$.
 - b. $\frac{21}{10} + \frac{7}{30}$.

Chapter XI

Answers for exercises

1. $\frac{3}{8}$

2. Yes (GCF(25,14) = 1)

3. a. 2×71

b. $3 \times 5 \times 5 \times 11$

c. $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

4. 27 $216 \div 81 = 2$, remainder 54
 $81 \div 54 = 1$, remainder 27
 $54 \div 27 = 2$, remainder 0

5. 648

6. a. $\frac{7}{15}$ work: $\frac{2}{15} + \frac{5}{15} = \frac{2+5}{15} = \frac{7}{15}$

b. $\frac{25}{36}$ work: $\frac{1}{4} + \frac{4}{9} = \frac{9}{36} + \frac{16}{36} = \frac{25}{36}$

7. a. $\frac{2}{8}$, or $\frac{1}{4}$ work: $\frac{7}{8} - \frac{5}{8} = \frac{7-5}{8} = \frac{2}{8} = \frac{1}{4}$

b. $\frac{3}{25}$ work: $\frac{23}{25} - \frac{4}{5} = \frac{23}{25} - \frac{20}{25} = \frac{23-20}{25} = \frac{3}{25}$

8. $\frac{2}{3}$

9. a. 3 work: $\frac{12}{15} + \frac{4}{15} = 12 \div 4 = 3$

b. $\frac{15}{33}$ work: $\frac{3}{7} + \frac{33}{35} = \frac{15}{35} + \frac{33}{35} = 15 \div 33 = \frac{15}{33}$

10. a. $\frac{4}{3}$, or $1\frac{1}{3}$ work: $\frac{3}{4} + \frac{9}{16} = \frac{3}{4} \times \frac{4}{4} + \frac{9}{16} = \frac{12}{16} + \frac{9}{16} = \frac{21}{16}$

b. 9 work: $\frac{21}{10} + \frac{7}{30} = \frac{21}{10} \times \frac{3}{3} + \frac{7}{30} = \frac{63}{30} + \frac{7}{30} = \frac{70}{30} = 9$

Chapter XII

Decimals

PART I: Help to You in Learning Mathematics

Decimals: The numerals in the box at the right are *decimals*.

The dot in a decimal is called the *decimal point*. As with standard numerals for whole numbers, a digit *in its place* indicates a number which is the product of the number named by the digit and the value of the place. For 2.3, the 2 is in one's place and indicates 2×1 , or 2, while the 3 is in tenth's place and indicates $3 \times \frac{1}{10}$, or $\frac{3}{10}$.

2.3
15.674
0.0037

The first place shown to the right of the decimal point is tenth's place, the next is hundredth's place, the next is thousandth's place, the next is ten-thousandth's place, and so on.

The decimals 2.3, 15.674, and 0.0037 may be read *two and three tenths*, *fifteen and six hundred seventy-four thousandths*, and *thirty-seven ten-thousandths*, respectively. Note the reading of "and" for the decimal point.

"Rounding" a number: Before considering rounding a number named with a decimal, let us review rounding whole numbers named with standard numerals.

At the right, 42 is shown nearer 40 than 50. This suggests that 42 *rounded to the nearest ten* is 40. Another way to round to the nearest ten is to consider the digit in



one's place. If that digit is 0, 1, 2, 3, or 4, we round to the lesser multiple of 10. If the digit is 5, 6, 7, 8, or 9, we round to the greater multiple of 10. To round to the nearest hundred, we consider the digit in ten's place; to round to the nearest thousand, we consider the digit in hundred's place; and so on.

Examples:

465 rounded to the nearest ten is 470.

2,307 rounded to the nearest hundred is 2,300.

47,894 rounded to the nearest thousand is 48,000.

Now, let's round numbers named with decimals. We round as with whole numbers named with standard numerals. For example, to round 2.78 to the nearest

tenth, we consider the digit in hundredth's place. Since that digit is 8, we round 2.78 to 2.8. If the digit in hundredth's place had been 0, 1, 2, 3, or 4, the number would have been rounded to 2.7.

Examples:

5.643 rounded to the nearest hundredth is 5.64.

0.2735 rounded to the nearest thousandth is 0.274.

Adding with decimals: Study the work shown at the right for finding $2.47 + 3.85$.

Add hundredths: $5 + 7 = 12$. Rename 12 hundredths as 1 tenth + 2 hundredths. Write a 2 in hundredth's place in the numeral for the sum and write a 1 in the column that shows tenths.

$$\begin{array}{r} 2.47 \\ + 3.85 \\ \hline 6.32 \end{array}$$

Add tenths: $8 + 4 + 1(\text{carry}) = 13$. Rename 13 tenths as 1 one + 3 tenths. Write a 3 in tenth's place in the numeral for the sum and write a 1 in the column that shows ones.

Add ones: $3 + 2 + 1(\text{carry}) = 6$. Write a 6 in one's place in the numeral for the sum.

Therefore, $2.47 + 3.85 = 6.32$.

Subtracting with decimals: The work shown at the right shows how to find $5.4 - 1.8$ by using the decomposition method. Recall that this method involves renaming the sum (minuend).

$$\begin{array}{r} 4 \quad 14 \\ \cancel{5.4} \\ - 1.8 \\ \hline 3.6 \end{array}$$

Subtract tenths: We cannot subtract 8 tenths from 4 tenths, so we rename 5.4 as 4 ones + 14 tenths. $14 - 8 = 6$, so we write a 6 in tenth's place.

Subtract ones: $4 - 1 = 3$, so we write a 3 in one's place.

Therefore, $5.4 - 1.8 = 3.6$.

Multiplying with decimals: Multiplying with decimals is similar to multiplying whole numbers named with standard numerals. One difficulty, however, is the placement of the decimal point in the numeral for the product.

Look at the work shown at the right for finding 0.7×2.54 . To properly place the decimal point in the numeral for the product, think: *tenths times hundredths is thousandths* ($\frac{1}{10} \times \frac{1}{100} = \frac{1}{1000}$), so the decimal for the product will show thousandths.

$$\begin{array}{r} 2.54 \\ \times 0.7 \\ \hline 1.778 \end{array}$$

Also, to properly place the decimal point in the numeral for the product, we may use the following rule. If the decimal for one factor shows m decimal places and the decimal for the other factor shows n decimal places, then the decimal for the product will show $m + n$ decimal places. For example, for 0.7×2.54 , 0.7 shows 1 decimal place and 2.54 shows 2 decimal places, so the decimal for the product will show $1 + 2$, or 3, decimal places.

Dividing with decimals: Study the work shows at the right for finding $2.05 \div 5$. Think of 2.05 as 20 tenths $\div 5$ hundredths.

$$\begin{array}{r} .41 \\ 5 \overline{)2.05} \\ \underline{20} \\ 5 \\ \underline{5} \\ 0 \end{array}$$

Divide tenths: $20 \div 5 = 4$, so 20 tenths $\div 5 = 4$ tenths.

Write a 4 in tenth's place in the numeral for the quotient.

Divide hundredths: $5 \div 5 = 1$, so 5 hundredth's $\div 5 = 1$ hundredth. Write a 1 in hundredth's place in the numeral for the quotient.

Therefore, $2.05 \div 5 = .41$ (or 0.41).

Now study this work for finding $0.483 \div 2.3$. Recall the principle

$$\begin{array}{r} .21 \\ 2.3 \overline{)0.483} \\ \underline{46} \\ 23 \\ \underline{23} \\ 0 \end{array} = \begin{array}{r} .21 \\ 23 \overline{)4.83} \\ \underline{46} \\ 23 \\ \underline{23} \\ 0 \end{array}$$

of compensation -- if both the product (dividend) and the known factor (divisor) are multiplied by the same nonzero number, then the

unknown factor (quotient) remains unchanged. Using this principle, we multiply both 0.483 , the product, and 2.3 , the known factor, by 10 so that the unknown factor can be found by dividing by a whole number.

In general, when the divisor is named with a decimal, we may multiply both the dividend and the divisor by a power of 10 so that the quotient can be found by dividing by a whole number.

Examples:

For $28.625 \div 3.26$, multiply by 10^2 , or 100:

$$3.26 \overline{)28.625} = 326 \overline{)2862.5}$$

For $46 \div 0.023$, multiply by 10^3 , or 1,000:

$$0.023 \overline{)46} = 23 \overline{)46000}$$

Renaming rational numbers: Suppose a rational number is named with a fraction. To rename the number with a decimal, we may think of the fraction as indicating a division and then do the division work. For example, consider the division work started at the right for renaming $\frac{2}{3}$ with a decimal. If we continue the division work, we will keep getting 6's. So, a decimal for $\frac{2}{3}$ is $0.66\ldots$. To indicate that "6" repeats endlessly, we write a bar over the 6. That is, we write $0.6\overline{6}$. A decimal such as $0.6\overline{6}$ is a *repeating decimal*. In a repeating decimal, there is a block of digits which repeats endlessly. $0.3\overline{3}$, the decimal for $\frac{1}{3}$ is a repeating decimal -- "3" is the repeating block. $0.63\overline{63}$, the decimal for $\frac{7}{11}$, is a repeating decimal -- "63" is the repeating block.

$$\begin{array}{r} 0.66 \\ 3 \overline{) 2.00} \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

If we continue this division work for renaming $\frac{1}{2}$ with a decimal, we will keep getting 0's. So a repeating decimal for $\frac{1}{2}$ is $0.50\overline{0}$ where "0" is the repeating block. However, instead of writing $0.50\overline{0}$, we simply write 0.5. Such a repeating decimal is traditionally called a *terminating, repeating decimal* or simply, a *terminating decimal* while a decimal such as $0.63\overline{63}$ is called a *non-terminating, repeating decimal*.

$$\begin{array}{r} 0.50 \\ 2 \overline{) 1.00} \\ \underline{10} \\ 0 \\ \underline{0} \end{array}$$

A repeating decimal (either non-terminating or terminating) names a rational number. Also, every rational number may be named with a repeating decimal. Recall that numbers such as $\sqrt{2}$ and π are not rational numbers; they are irrational numbers. Decimals for irrational numbers are not repeating decimals.

Now suppose a rational number is named with a decimal. Let us consider how to rename the number with a fraction. If the decimal is a terminating decimal, then we can easily rename with a fraction showing a denominator that is a power of 10. (Then, if necessary, we can rename with a fraction in simplest form.)

Examples:

For 0.3, think *three tenths*, so a fraction for 0.3 is $\frac{3}{10}$.

For 0.24, think *twenty-four hundredths*, so a fraction for 0.24 is $\frac{24}{100}$.

For 2.607, think *two thousand six hundred seven thousandths*, so a fraction for 2.607 is $\frac{2607}{1000}$.

If the decimal is a non-terminating decimal, then the following examples illustrate the procedure that may be used to rename with a fraction.

Example 1: Rename $0.\overline{33}$ with a fraction.

Let $A = 0.\overline{33}$. Then $10A = 3.\overline{33}$. Subtracting, we have

$$10A = 3.\overline{33}$$

$$\underline{A = 0.\overline{33}}$$

$$9A = 3$$

$$\text{So, } A = \frac{3}{9}, \text{ or } \frac{1}{3}.$$

Example 2: Rename $0.\overline{6363}$ with a fraction.

Let $A = 0.\overline{6363}$. Then $100A = 63.\overline{6363}$. Subtracting, we have

$$100A = 63.\overline{6363}$$

$$\underline{A = 0.\overline{6363}}$$

$$99A = 63$$

$$\text{So, } A = \frac{63}{99}, \text{ or } \frac{7}{11}.$$

Example 3: Rename $3.\overline{6215215}$ with a fraction.

Let $A = 3.\overline{6215215}$. Then $1000A = 3621.\overline{5215215}$. Subtracting, we have

$$1000A = 3621.\overline{5215215}$$

$$\underline{A = 3.\overline{6215215}}$$

$$999A = 3617.9$$

$$\text{Thus, } 9990A = 36179$$

$$\text{So, } A = \frac{36179}{9990}$$

Notice that, in each case, A is multiplied by 10^n where n is the number of digits in the block of digits which repeats endlessly.

Chapter XII

Exercises

1. Write a decimal for the number.
 - a. Fifteen and forty-three hundredths
 - b. Twenty-nine thousandths
2. Round to the nearest hundredth.
 - a. 5.062
 - b. 0.0557
 - c. 89.008
3. Rename with a fraction in simplest form.
 - a. 0.07
 - b. 0.125
 - c. 0.0004
4. Rename with a decimal
 - a. $\frac{9}{100}$
 - b. $\frac{23}{25}$
 - c. $\frac{7}{500}$
5. Copy and add.

a. 5.037 <u>0.786</u>	b. 0.0928 0.0032 <u>.0005</u>	c. 15.609 4.337 <u>8.943</u>
--------------------------	-------------------------------------	------------------------------------
6. Copy and subtract.

a. 3.047 <u>0.062</u>	b. 0.9012 <u>0.0987</u>	c. 43.60 <u>5.09</u>
--------------------------	----------------------------	-------------------------
7. The decimal for the product of 13.406 and 7.39 will show how many decimal places?
8. Find the unknown factor to the nearest thousandth.
 - a. $3.68 \div 0.9$
 - b. $0.076 \div 0.32$
9. Rename with a decimal. If the decimal is non-terminating, write a bar over the block of digits which repeats endlessly.
 - a. $\frac{25}{99}$
 - b. $\frac{6}{7}$
 - c. $\frac{127}{3}$
10. Rename with a fraction in simplest form.
 - a. $0.72\overline{72}$
 - b. $5.46\overline{6}$
 - c. $13.025\overline{025}$

Chapter XII

Answers for exercises

- | | | |
|--|--------------------|------------------------|
| 1. a. 15.43 | b. 0.029 | |
| 2. a. 5.06 | b. 0.06 | c. 89.01 |
| 3. a. $\frac{7}{100}$ | b. $\frac{1}{8}$ | c. $\frac{1}{2500}$ |
| 4. a. 0.09 | b. 0.92 | c. 0.014 |
| 5. a. 5.823 | b. 0.0965 | c. 28.889 |
| 6. a. 2.985 | b. 0.8025 | c. 38.51 |
| 7. 5 | | |
| 8. a. 4.089 | b. 0.238 | |
| 9. a. $0.25\overline{25}$ (or $0.\overline{25}$) | | |
| b. $0.857142\overline{857142}$ (or $0.\overline{857142}$) | | |
| c. $42.3\overline{3}$ (or $42.\overline{3}$) | | |
| 10. a. $\frac{8}{11}$ | b. $\frac{82}{15}$ | c. $\frac{13012}{999}$ |

Chapter XII

Decimals

PART I: Help to You in Learning Mathematics

Decimals: The numerals in the box at the right are *decimals*.

The dot in a decimal is called the *decimal point*. As with standard numerals for whole numbers, a digit *in its place* indicates a number which is the product of the number named by the digit and the value of the place. For 2.3, the 2 is in one's place and indicates 2×1 , or 2, while the 3 is in tenth's place and indicates $3 \times \frac{1}{10}$, or $\frac{3}{10}$.

2.3
15.674
0.0037

The first place shown to the right of the decimal point is tenth's place, the next is hundredth's place, the next is thousandth's place, the next is ten-thousandth's place, and so on.

The decimals 2.3, 15.674, and 0.0037 may be read *two and three tenths*, *fifteen and six hundred seventy-four thousandths*, and *thirty-seven ten-thousandths*, respectively. Note the reading of "and" for the decimal point.

"Rounding" a number: Before considering rounding a number named with a decimal, let us review rounding whole numbers named with standard numerals.

At the right, 42 is shown nearer 40 than 50. This suggests that 42 *rounded to the nearest ten* is 40. Another way to round to



the nearest ten is to consider the digit in one's place. If that digit is 0, 1, 2, 3, or 4, we round to the lesser multiple of 10. If the digit is 5, 6, 7, 8, or 9, we round to the greater multiple of 10. To round to the nearest hundred, we consider the digit in ten's place; to round to the nearest thousand, we consider the digit in hundred's place; and so on.

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Adding with decimals: Study the work shown at the right for finding $2.47 + 3.85$.

Add hundredths: $5 + 7 = 12$. Rename 12 hundredths as 1 tenth + 2 hundredths. Write a 2 in hundredth's place in the numeral for the sum and write a 1 in the column that shows tenths.

$$\begin{array}{r} 2.47 \\ + 3.85 \\ \hline 6.32 \end{array}$$

Add tenths: $8 + 4 + 1(\text{carry}) = 13$. Rename 13 tenths as 1 one + 3 tenths. Write a 3 in tenth's place in the numeral for the sum and write a 1 in the column that shows ones.

Add ones: $3 + 2 + 1(\text{carry}) = 6$. Write a 6 in one's place in the numeral for the sum.

Therefore, $2.47 + 3.85 = 6.32$.

Subtracting with decimals: The work shown at the right shows how to find $5.4 - 1.8$ by using the decomposition method. Recall that this method involves renaming the sum (minuend).

$$\begin{array}{r} 4 \quad 14 \\ 5.4 \\ - 1.8 \\ \hline 3.6 \end{array}$$

Subtract tenths: We cannot subtract 8 tenths from 4 tenths, so we rename 5.4 as 4 ones + 14 tenths. $14 - 8 = 6$, so we write a 6 in tenth's place.

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Also, to properly place the decimal point in the numeral for the product, we may use the following rule. If the decimal for one factor shows m decimal places and the decimal for the other factor shows n decimal places, then the decimal for the product will show $m + n$ decimal places. For example, for 0.7×2.54 , 0.7 shows 1 decimal place and 2.54 shows 2 decimal places, so the decimal for the product will show $1 + 2$, or 3, decimal places.

Dividing with decimals: Study the work shows at the right for finding $2.05 \div 5$. Think of 2.05 as 20 tenths $\div$ 5 hundredths.

$$\begin{array}{r} .41 \\ 5 \overline{)2.05} \\ \underline{20} \\ 5 \\ \underline{5} \\ 0 \end{array}$$

Divide tenths: $20 \div 5 = 4$, so 20 tenths $\div$ 5 = 4 tenths.

Write a 4 in tenth's place in the numeral for the quotient.

Divide hundredths: $5 \div 5 = 1$, so 5 hundredth's $\div$ 5 = 1 hundredth. Write a 1 in hundredth's place in the numeral for the quotient.

Therefore, $2.05 \div 5 = .41$ (or 0.41).

Now study this work for finding $0.483 \div 2.3$. Recall the principle

$$\begin{array}{r} 2.3 \overline{)0.483} \\ \underline{23} \\ 483 \\ \underline{46} \\ 23 \\ \underline{23} \\ 0 \end{array} = \begin{array}{r} .21 \\ 23 \overline{)4.83} \\ \underline{46} \\ 23 \\ \underline{23} \\ 0 \end{array}$$

of compensation -- if both the

product (dividend) and the known

factor (divisor) are multiplied by

the same nonzero number, then the

unknown factor (quotient) remains unchanged. Using this principle, we multiply both 0.483, the product, and 2.3, the known factor, by 10 so that the unknown factor can be found by dividing by a whole number.

In general, when the divisor is named with a decimal, we may multiply both the dividend and the divisor by a power of 10 so that the quotient can be found by dividing by a whole number.

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Renaming rational numbers: Suppose a rational number is named with a fraction. To rename the number with a decimal, we may think of the fraction as indicating a division and then do the division work. For example, consider the division work started at the right for renaming

$\frac{2}{3}$ with a decimal. If we continue the division work, we will keep getting 6's. So, a decimal for $\frac{2}{3}$ is $0.66\ldots$.

To indicate that "6" repeats endlessly, we write a bar over the 6. That is, we write $0.6\overline{6}$. A decimal such as $0.6\overline{6}$ is a *repeating decimal*. In a repeating decimal, there

is a block of digits which repeats endlessly. $0.3\overline{3}$, the decimal for $\frac{1}{3}$ is a repeating decimal -- "3" is the repeating block. $0.63\overline{63}$, the decimal for $\frac{7}{11}$, is a repeating decimal -- "63" is the repeating block.

$$\begin{array}{r} 0.66 \\ 3 \overline{) 2.00} \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

If we continue this division work for renaming $\frac{1}{2}$ with a decimal, we will keep getting 0's. So a repeating decimal for $\frac{1}{2}$ is

$0.50\overline{0}$ where "0" is the repeating block. However, instead of writing $0.50\overline{0}$, we simply write 0.5.

Such a repeating decimal is traditionally called a *terminating, repeating decimal* or simply, a *terminating decimal* while a decimal such as $0.63\overline{63}$ is called a *non-terminating, repeating decimal*.

$$\begin{array}{r} 0.50 \\ 2 \overline{) 1.00} \\ \underline{10} \\ 0 \\ \underline{0} \end{array}$$

A repeating decimal (either non-terminating or terminating) names a rational number. Also, every rational number may be named with a repeating decimal. Recall that numbers such as $\sqrt{2}$ and π are not rational numbers; they are irrational numbers. Decimals for irrational numbers are not repeating decimals.

Now suppose a rational number is named with a decimal. Let us consider how to rename the number with a fraction. If the decimal is a terminating decimal, then we can easily rename with a fraction showing a denominator that is a power of 10. (Then, if necessary, we can rename with a fraction in simplest form.)

Examples:

For 0.3, think *three tenths*, so a fraction for 0.3 is $\frac{3}{10}$.

For 0.24, think *twenty-four hundredths*, so a fraction for 0.24 is $\frac{24}{100}$.

For 2.607, think *two thousand six hundred seven thousandths*, so a fraction for 2.607 is $\frac{2607}{1000}$.

If the decimal is a non-terminating decimal, then the following examples illustrate the procedure that may be used to rename with a fraction.

Example 1: Rename $0.\overline{33}$ with a fraction.

Let $A = 0.\overline{33}$. Then $10A = 3.\overline{33}$. Subtracting, we have

$$\begin{array}{r} 10A = 3.\overline{33} \\ A = 0.\overline{33} \\ \hline 9A = 3 \end{array}$$

$$\text{So, } A = \frac{3}{9}, \text{ or } \frac{1}{3}.$$

Example 2: Rename $0.\overline{6363}$ with a fraction.

Let $A = 0.\overline{6363}$. Then $100A = 63.\overline{6363}$. Subtracting, we have

$$\begin{array}{r} 100A = 63.\overline{6363} \\ A = 0.\overline{6363} \\ \hline 99A = 63 \end{array}$$

$$\text{So, } A = \frac{63}{99}, \text{ or } \frac{7}{11}.$$

Example 3: Rename $3.\overline{6215215}$ with a fraction.

Let $A = 3.\overline{6215215}$. Then $1000A = 3621.\overline{5215215}$. Subtracting, we have

$$\begin{array}{r} 1000A = 3621.\overline{5215215} \\ A = 3.\overline{6215215} \\ \hline 999A = 3617.9 \end{array}$$

$$\text{Thus, } 9990A = 36179$$

$$\text{So, } A = \frac{36179}{9990}$$

Notice that, in each case, A is multiplied by 10^n where n is the number of digits in the block of digits which repeats endlessly.

Chapter XII

Exercises

1. Write a decimal for the number.
 - a. Fifteen and forty-three hundredths
 - b. Twenty-nine thousandths
2. Round to the nearest hundredth.
 - a. 5.062
 - b. 0.0557
 - c. 89.008
3. Rename with a fraction in simplest form.
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 - c. 0.0004
4. Rename with a decimal
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a. 5.037 <u>0.786</u>	b. 0.0928 0.0032 <u>.0005</u>	c. 15.609 4.337 <u>8.943</u>
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a. 3.047 <u>0.062</u>	b. 0.9012 <u>0.0987</u>	c. 43.60 <u>5.09</u>
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7. The decimal for the product of 13.406 and 7.39 will show how many decimal places?
8. Find the unknown factor to the nearest thousandth.
 - a. $3.68 \div 0.9$
 - b. $0.076 \div 0.32$
9. Rename with a decimal. If the decimal is non-terminating, write a bar over the block of digits which repeats endlessly.
 - a. $\frac{25}{99}$
 - b. $\frac{6}{7}$
 - c. $\frac{127}{3}$
10. Rename with a fraction in simplest form.
 - a. $0.72\overline{72}$
 - b. $5.46\overline{6}$
 - c. $13.025\overline{025}$

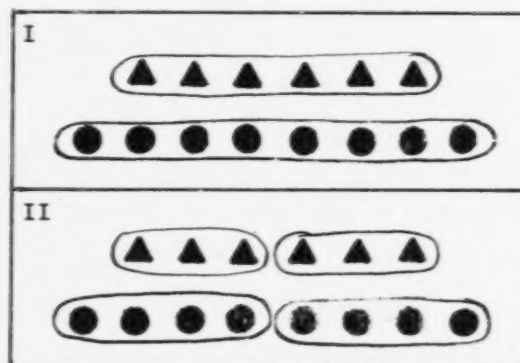
Chapter XII

Answers for exercises

- | | | |
|--|--------------------|------------------------|
| 1. a. 15.43 | b. 0.029 | |
| 2. a. 5.06 | b. 0.06 | c. 89.01 |
| 3. a. $\frac{7}{100}$ | b. $\frac{1}{8}$ | c. $\frac{1}{2500}$ |
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| 5. a. 5.823 | b. 0.0965 | c. 28.889 |
| 6. a. 2.985 | b. 0.8025 | c. 38.51 |
| 7. 5 | | |
| 8. a. 4.089 | b. 0.238 | |
| 9. a. $0.25\overline{25}$ (or $0.\overline{25}$) | | |
| b. $0.857142\overline{857142}$ (or $0.\overline{857142}$) | | |
| c. $42.\overline{33}$ (or $42.\overline{3}$) | | |
| 10. a. $\frac{8}{11}$ | b. $\frac{82}{15}$ | c. $\frac{13012}{999}$ |

PART I: Help to You in Learning Mathematics

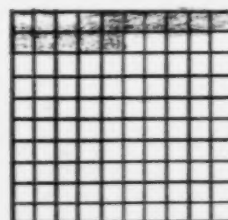
Ratio: In Box I are shown 6 triangular regions and 8 circular regions. The *ratio* of the number of triangular regions to the number of circular regions is 6 to 8. To express this, we may write 6 to 8 (or 6:8 or $\frac{6}{8}$ or (6, 8)). In Box II, both the set of triangular regions and the set of circular regions are shown partitioned into equivalent subsets. (Recall that



two finite sets are equivalent if each set has the same number of elements.) For every 3 triangular regions there are 4 circular regions. This is a ratio of 3 to 4. Therefore, for the regions shown in Box I, the ratio of the number of triangular regions to the number of circular regions may be expressed as 3 to 4 (or 3:4 or $\frac{3}{4}$ or (3,4)). Note that the ratio may be expressed either with the fraction $\frac{6}{8}$ or with the fraction $\frac{3}{4}$.

Proportion: In the last section, we found that $\frac{6}{8}$ and $\frac{3}{4}$ each express the same ratio. To show this, we write $\frac{6}{8} = \frac{3}{4}$. A statement of equality of ratios is a *proportion*. So, $\frac{6}{8} = \frac{3}{4}$ (which may be read *six is to eight as three is to four*) is a proportion. A mathematical sentence showing the relationship expressed in a problem is sometimes in the form of a proportion involving an unknown (e.g. $\frac{3}{10} = \frac{15}{n}$). To solve a proportion such as $\frac{3}{10} = \frac{15}{n}$, we may think $\frac{3}{10} = \frac{15}{n}$ if and only if $3 \times n = 10 \times 15$. For $\frac{3}{10} = \frac{15}{n}$, $3n = 150$ and therefore, $n = 50$.

Per cent: A ratio which compares the number of members in one set to the number of members in a set containing 100 members is called a *per cent*. The diagram at the right shows 100 small congruent regions, 15 of which are shaded. The ratio of the number of shaded



regions to the total number of regions is 15 to 100. So, we may say that $\frac{15}{100}$ of the diagram is shaded or that 15% (read *fifteen per cent*) of the diagram is shaded. A form such as 15% is called a *per cent form*.

Expressing ratios with per cent forms: To express a ratio with a per cent form, first express the ratio with a fraction showing denominator 100. Then write the numeral for the numerator and the symbol "%." For example, the ratio 2 to 5 may be expressed as $\frac{2}{5}$ and $\frac{2}{5} = \frac{40}{100}$, so the ratio 2 to 5 may be expressed as 40%.

Three types of per cent problems: Consider the following situation.

One afternoon, Sue typed 25% of the 20 letters that she said she would type for a friend. That afternoon, Sue typed 5 letters.

A mathematical sentence which shows the relationship expressed in the above situation is $25\% \times 20 = 5$. The situation is based on the number of letters to be typed, so that number (20) is the *base factor*. The ratio or per cent for the situation is 25%, so 25% is the *ratio factor*. 5, the product of the ratio factor and the base factor refers, in this case, to the number of letters that Sue typed. This is summarized as follows:

RATIO FACTOR	BASE FACTOR	PRODUCT
Per cent for	Number of letters	Number of
letters typed	to be typed	letters typed
25%	20	5

Mathematical sentence: $25\% \times 20 = 5$

Now let's consider three types of per cent problems. In each type, either the ratio factor, the base factor, or the product is unknown.

(1) *The product is unknown:* Consider the following problem.

Elaine works in the school kitchen. She washed 80% of the 225 dishes that had to be washed. How many dishes did Elaine wash?

For this problem we have,

RATIO FACTOR	BASE FACTOR	PRODUCT
Per cent for	Number of dishes	Number of dishes
dishes washed	to be washed	washed
80%	225	n

Mathematical sentence: $80\% \times 225 = n$

Notice that n represents the unknown product. When the two factors are known, we multiply to find the product. To find $80\% \times 225$, we may express 80% with the fraction $\frac{80}{100}$ or with the fraction $\frac{4}{5}$ or with the decimal 0.8. Study the work shown in the following boxes.

$$\begin{aligned} n &= 80\% \times 225 \\ &= \frac{80}{100} \times 225 \\ &= \frac{4}{5} \times 225 \\ n &= 180 \end{aligned}$$

$$\begin{aligned} n &= 80\% \times 225 \\ n &= 0.8 \times 225 \\ &\quad 225 \\ &\quad \times 0.8 \\ &\quad \hline &\quad 180.0 \\ n &= 180 \end{aligned}$$

Elaine washed 180 dishes.

(2) *The ratio factor is unknown:* Consider the following problem.

Miss Sutton gave a 20-question mathematics test to her class. Seven of the questions were geometry questions. What per cent of the 20 questions were geometry questions?

For this problem we have,

RATIO FACTOR
Per cent for
geometry questions

BASE FACTOR
Number of
questions

PRODUCT
Number of
geometry
questions

n

20

7

Mathematical sentence: $n \times 20 = 7$

Notice that n represents the ratio factor. When the product and one factor are known, we divide to find the unknown factor. Study the work shown in the box. To express $\frac{7}{20}$ as a per cent, think $\frac{7}{20} = \frac{35}{100}$, so $\frac{7}{20} = 35\%$.

$$\begin{aligned} n \times 20 &= 7 \\ n &= 7 \div 20 \\ &= \frac{7}{20}, \text{ or } 35\% \end{aligned}$$

35% of the questions were geometry questions.

(3) *The base factor is unknown:* Consider the following problem.

Of the letters that Diana received one day, 75% were letters from her boyfriend. She received 3 letters from her boyfriend. In all, how many letters did she receive that day?

For this problem we have,

RATIO FACTOR
Per cent for letters
from her boyfriend
75%

BASE FACTOR
Number of
letters
n

PRODUCT
Number of letters
from boyfriend
3

Mathematical sentence: $75\% \times n = 3$

Notice that n represents the base factor. When the product and one factor are known, we divide to find the unknown factor. Study the work shown in the following boxes.

$$\begin{aligned} 75\% \times n &= 3 \\ n &= 3 \div 75\% \\ &= 3 \div \frac{3}{4} \\ &= 3 \times \frac{4}{3} \\ n &= 4 \end{aligned}$$

$$\begin{aligned} 75\% \times n &= 3 \\ n &= 3 \div 75\% \\ n &= 3 \div 0.75 \\ 0.75 \overline{)3} &= 75 \overline{)300} \begin{array}{r} 4. \\ 300 \\ \hline \end{array} \\ n &= 4 \end{aligned}$$

Diana received 4 letters that day.

Chapter XIII

Exercises

1. Express the ratio with a fraction in simplest form.

a. 17 to 50

b. 20:50

c. 140:161

2. Write a per cent form to express what part of this diagram is shaded.



3. Express the ratio with a per cent form.

a. 1 to 50

b. 23:25

c. 2:200

4. Rename with a fraction in simplest form.

a. 60%

b. 45%

c. 4%

5. Solve.

a. $\frac{3}{14} = \frac{n}{98}$

b. $\frac{20}{550} = \frac{4}{n}$

6. For
- $24\% \times n = 48$
- , which is unknown,
- the ratio factor or the base factor?*

7. Solve. Express the answer with a per cent form.

a. $n \times 300 = 48$

b. $15 = n \times 125$

For each of examples 8 - 10, write and solve a mathematical sentence to find the answer for the problem.

8. Walpole High won 21 of 25 basketball games. What per cent of the 25 games did they win?
9. The attendance at one basketball game was only 55% of the seating capacity of the gym. The gym seats 540. What was the attendance at that game?
10. During one basketball game, Eric made baskets on 60% of the shots that he took. He made 9 baskets. How many shots did he take?

Chapter XIII

Answers for Exercises

- | | | |
|---|-------------------|--------------------|
| 1. a. $\frac{17}{50}$ | b. $\frac{2}{5}$ | c. $\frac{20}{23}$ |
| 2. 25% | | |
| 3. a. 2% | b. 92% | c. 1% |
| 4. a. $\frac{3}{5}$ | b. $\frac{9}{20}$ | c. $\frac{1}{25}$ |
| 5. a. $n = 21$ | b. $n = 110$ | |
| 6. the base factor | | |
| 7. a. $n = 16\%$ | b. $n = 12\%$ | |
| 8. $n \times 25 = 21$; 84% | | |
| 9. $55\% \times 540 = n$; 297 (people) | | |
| 10. $60\% \times n = 9$; 15 shots | | |